

“The Classical Limit” Is Not One Limit

— Equations, Spectrum, Wavefunction, State: the Separator Is What Is Asked to Become Classical — [Paper 180]

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— *From thinking to syncing.* —

ABSTRACT

Paper 113 wrote that quantum mechanics is a deformation of classical mechanics by $\hbar$, and that $\hbar \rightarrow 0$ returns the classical algebra. That is a statement about structures. This paper asks the neighbouring question — does the behaviour return? They are different sentences, and the second has several answers. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the equations. $\langle V'(x) \rangle = V'(\langle x \rangle)$ is exact only for V at most quadratic. At cubic order the gap is σ^2 and at quartic order exactly $3\mu\sigma^2$ (Section 2). Second, the spectrum. In hydrogen the ratio of the $n \rightarrow n-1$ frequency to the classical orbital frequency runs 3.0, 1.40625, 1.0015020025, approaching 1 like $1 + 3/(2n)$. Not once does $\hbar$ appear in that ratio — this is a statement about large quantum number, not about small $\hbar$ (Section 3). Third, the wavefunction. The same Bohr–Sommerfeld recipe is exact at every n for the oscillator (error 0) and never exact for the quartic, improving from 1.8×10^{-1} to 2.2×10^{-5} with n (Section 4). Fourth, and this is the core. The limits $\hbar \rightarrow 0$ and $t \rightarrow \infty$ do not commute. Holding $t = 2$, the difference dies with $\hbar$: $0.1762 \rightarrow 0.1050 \rightarrow 0.0581 \rightarrow 0.0307$. Holding $\hbar = 0.1$ and waiting, the swing collapses from 1.5 to 0.000195 and then comes back to 0.664 at $t = 199.85$ — while the classical orbit has not moved from 1.4986 (Section 5). Fifth, the survival time is finite all the same. It grows $8.16 \rightarrow 10.64 \rightarrow 14.28 \rightarrow 19.20$, and only as a power law, near $\hbar^{-0.41}$ (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED — Ehrenfest's theorem, Bohr's correspondence principle, the WKB approximation, and the collapse and revival of wavepackets are all standard. No interpretation of quantum mechanics is judged. On Paper 113: that paper showed the algebra returns; this one asks whether the behaviour does — a brake, not a refutation. On Paper 154: the pattern “two limits do not commute” is already that paper's subject (the Curie–Weiss magnet). Section 5 here is a second instance, not a new pattern — what is added is that the pattern also appears at the quantum-classical boundary. What is added is setting four limits against one separator, writing the Ehrenfest gap in closed form and confirming its proportionality to $\hbar$ in numbers, measuring the rate of the correspondence principle, placing side by side one WKB recipe that is exact in one well and asymptotic in another, and measuring the collapse and revival to a precision that survives a doubling of the basis.

1. Notation and position

Throughout, $H = p^2/2 + V(x)$ with unit mass. Every number here was computed for this paper, and no matrix result is quoted unless **doubling the basis left it alone** (values that moved are not reported). The classical orbits were integrated with my own Runge–Kutta scheme.

Position: Paper 113 wrote that modules are limits rather than walls, and joined the quantum to the classical by a deformation in $\hbar$. This paper stands next to it — the algebra returning and the behaviour returning are different sentences.

2. The equations: where Ehrenfest stops being exact

Observation 2.1

The rate of change of $\langle p \rangle$ is **exactly** $-\langle V'(x) \rangle$, while the classical equation reads $-V'(\langle x \rangle)$. These agree exactly when V' is linear, that is when V is at most quadratic.

Verification 2.2: With a Gaussian packet of mean $\mu = 0.7$ and standard deviation $\sigma = 0.35$.

$V(x)$	$V'(x)$	gap	predicted
$x^2/2$	x	0	0
$x^3/3$	x^2	0.1225	$\sigma^2 = 0.1225$
$x^4/4$	x^3	0.25725	$3\mu\sigma^2 = \mathbf{0.25725}$

(QED)

Observation 2.3

What makes the gap is the width of the packet, and the width is where $\hbar$ enters — for a minimum-uncertainty packet $\sigma^2 = \hbar/2$, so the gap is proportional to $\hbar$.

$\hbar$	$\sigma^2 = \hbar/2$	gap $3\mu\sigma^2$
1	0.5	1.05
0.01	0.005	0.0105
0.0001	0.00005	0.000105

(QED)

Observation 2.4

So at a fixed time the classical equation is recovered as $\hbar \rightarrow 0$. That proviso is worth remembering — Section 5 shows the same statement failing when the time is not held fixed.

3. The spectrum: Bohr's own correspondence, and its rate

Observation 3.1

Bohr (1913, 1920) said that for large quantum number the frequency radiated on $n \rightarrow n - 1$ approaches the classical orbital frequency. In Rydberg units hydrogen has $E_n = -1/n^2$, so the transition frequency is $1/(n - 1)^2 - 1/n^2$ while the classical orbital frequency at level n is $2/n^3$.

Verification 3.2

n	quantum ($n \rightarrow n - 1$)	classical ($2/n^3$)	ratio
2	0.75	0.25	3.0
5	0.0225	0.016	1.40625
10	0.00234567901235	0.002	1.17283950617
100	$2.03040506071 \times 10^{-6}$	2.0×10^{-6}	1.01520253035
1000	$2.00300400501 \times 10^{-9}$	2.0×10^{-9}	1.0015020025
10000	$2.00030004001 \times 10^{-12}$	2.0×10^{-12}	1.00015002

(QED)

Verification 3.3: The approach goes like $1 + 3/(2n)$.

n	ratio -1	$3/(2n)$
10	0.1728395062	0.15
100	0.01520253035	0.015
1000	0.001502002503	0.0015
10000	0.0001500200025	0.00015

(QED)

Observation 3.4

Nowhere in that ratio does $\hbar$ appear. The correspondence principle is not a statement that $\hbar$ is small but that the quantum number is large — a different sentence from the $\hbar \rightarrow 0$ of Section 2.

Verification 3.5: Nor is it about a single frequency. The jump $n \rightarrow n - k$ corresponds to the k th harmonic of the orbit.

transition ($n = 1000$)	classical side	ratio
$n \rightarrow n - 1$	fundamental	1.0015020025
$n \rightarrow n - 2$	second harmonic	1.00300802005
$n \rightarrow n - 3$	third harmonic	1.00451806774

What corresponds is a whole Fourier series. (QED)

4. The wavefunction: one recipe, exact and asymptotic

Observation 4.1

Take the Bohr–Sommerfeld condition with the added $1/2$ (the Maslov index) and apply it in the same form to two wells.

Verification 4.2: The oscillator $H = p^2/2 + x^2/2$.

n	WKB	matrix eigenvalue	error
0	0.5	0.5000000000000000	0
1	1.5	1.5000000000000000	0
5	5.5	5.5000000000000000	0
20	20.5	20.5000000000000000	0

Exact at every n , the ground state included. (QED)

Verification 4.3: The quartic well $H = p^2/2 + x^4$. Raising the basis through 100, 200, 400 moves $E_0 = 0.66798625915578$ by only 10^{-14} .

n	WKB	matrix eigenvalue	relative error
0	0.54626732507809	0.667986259156	1.822×10^{-1}
1	2.3635614446006	2.393644016466	1.257×10^{-2}
5	13.363764206125	13.379336552631	1.164×10^{-3}
10	31.649312728425	31.659456477223	3.204×10^{-4}
40	191.44847701265	191.452604854688	2.156×10^{-5}

(QED)

Observation 4.4

“Semiclassical” is not one condition either. In one well the recipe is exact at every n ; in the other it is only asymptotic in n . The same recipe means different things in different wells.

5. The core: $\hbar \rightarrow 0$ and $t \rightarrow \infty$ do not commute

Observation 5.1

Place a packet at rest at $x_0 = 1.5$ in the quartic well. On the classical side $\ddot{x} = -4x^3$ is integrated with my own Runge–Kutta scheme.

Verification 5.2: Hold the time at $t = 2$ and lower $\hbar$.

$\hbar$	quantum $\langle x \rangle(2)$	classical $x(2)$	difference
0.4	0.640102912023	0.463905500086	1.762×10^{-1}
0.2	0.568908324699	0.463905500086	1.050×10^{-1}
0.1	0.521994255976	0.463905500086	5.809×10^{-2}
0.05	0.494576155187	0.463905500086	3.067×10^{-2}

Halving $\hbar$ halves the difference, in agreement with the closed form of Section 2. Every row moved by only 10^{-14} when the basis was doubled. (QED)

Verification 5.3: Now hold $\hbar = 0.1$ and wait.

t	quantum swing	classical swing
0	1.500000	1.500000
10	0.482629	1.499910
20	0.014062	1.499426
40	0.001587	1.499486
120	0.022298	1.498760
200	0.664054	1.498946
240	0.289268	1.498613

The deepest collapse was **0.000195** at $t = 27.50$ and the strongest revival **0.664054** at $t = 199.85$. (QED)

Observation 5.4

The swing comes back. The classical orbit neither collapses nor revives and has not moved from 1.4986. So this is not a small correction to an orbit. Both the collapse and the revival come from the level spacing not being uniform — and nothing on the classical side answers to that.

Observation 5.5

Side by side.

order taken	result
fix t , then $\hbar \rightarrow 0$	classical (Ver. 5.2)
fix $\hbar$, then $t \rightarrow \infty$	not classical (Ver. 5.3)

This pattern is not new to this corpus. Paper 154 takes “two limits do not commute” as its subject and sets 0 against 0.957504 in the Curie–Weiss magnet. All that is added here is that the same pattern appears at the quantum-classical boundary.

6. And yet the survival time is finite

Verification 6.1: Measure how long it takes for the swing to fall below a tenth of its initial value.

$\hbar$	survival time	moved on doubling the basis
0.4	8.160	4.5×10^{-11}
0.2	10.640	2.7×10^{-11}
0.1	14.280	1.4×10^{-11}
0.05	19.200	1.5×10^{-11}

(QED)

Observation 6.2

It grows, and only as a power law (the measured log-log slope is near $\hbar^{-0.41}$). Making $\hbar$ small postpones the classical orbit's failure; it does not prevent it.

Observation 6.3

“Ehrenfest time” is itself one name for two behaviours. The short $\log(1/\hbar)$ often quoted belongs to chaotic systems, where the spreading is exponential. The well here is integrable, and measurement gives a power law. The exponent above is a measurement on one packet in one well, not a derived exponent.

7. The census

Observation 7.1

what becomes classical	name	when it holds	number here
the equations	Ehrenfest	exact up to quadratic	$3\mu\sigma^2$
the spectrum	Bohr's correspondence	large n (no $\hbar$)	$1 + 3/(2n)$
the wavefunction	WKB, semiclassical	depends on the well	error 0 vs 2.2×10^{-5}
the state	decoherence	superposition goes	Papers 55, 138

Observation 7.2

These four are not a staircase of strength. Decoherence makes the state classical without making the equations classical (as Paper 55 wrote, a diagonal density matrix does not say **which** outcome occurs). Conversely the equations can be classical while superposition survives. What settles it is what is being asked to be classical.

8. Conclusion

1. Ehrenfest is exact only up to quadratic (Ver. 2.2): the gap is exactly $3\mu\sigma^2 = 0.25725$.
2. The gap is proportional to $\hbar$ (Obs. 2.3).
3. No $\hbar$ appears in the correspondence principle (Obs. 3.4): the ratio is $1 + 3/(2n)$.
4. One WKB recipe is exact at every n for the oscillator (Ver. 4.2) and asymptotic for the quartic (Ver. 4.3).

5. **The two limits do not commute** (Ver. 5.2, 5.3): **the difference dies with $\hbar$, yet waiting collapses the swing to 0.000195 and returns it to 0.664.**

6. **The survival time grows and stays finite** (Ver. 6.1): **8.16 $\rightarrow$ 19.20.**

Closing: When someone says “take the classical limit”, the next question is what is being asked to become classical. The equations, the spectrum, the wavefunction, or the state — all four go by the same phrase, and they hold under different conditions and fail in different places. And the algebra returning does not mean the behaviour returns.

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