

The Multiplication Does Not Decide the Second System

— One Algebra Carrying Two Coproducts, and the Tensor Product of Representations Changes — [Paper 181]

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— *From thinking to syncing.* —

ABSTRACT

Paper 125 touched a Hopf algebra once, only to say that it is an object unrelated to the Hopf fibration. It never said what one does. This paper does — the multiplication says how a symmetry acts on ONE system and the coproduct says how to build a second, and the multiplication does not decide the second. **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, two group algebras that are the same as algebras. Both $\mathbb{C}[\mathbb{Z}/4]$ and $\mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2]$ are four dimensional and commutative, and both character tables have determinant 16 — each is $\mathbb{C}^4$ (Section 2). Second, and this is the core. Their tensor products of representations differ. Under tensoring, the orders of the four characters are 1, 4, 2, 4 for $\mathbb{Z}/4$ and 1, 2, 2, 2 for $\mathbb{Z}/2 \times \mathbb{Z}/2$ — what the multiplication left undecided, the coproduct decides (Section 3). Third, this is why spins add while phases multiply. It is the difference between the grouplike $\Delta(g) = g \otimes g$ and the primitive $\Delta(x) = x \otimes 1 + 1 \otimes x$. Through the primitive one, J^2 has eigenvalues 0, 2, 2, 2 (Section 4). Fourth, the antipode is what gives conjugate representations (Section 5). Fifth, the fence is cocommutativity. At $q = 1$ braiding is exactly the flip and squares to one, while at $q = 1.3$ the quantity $\hat{R}^2 - I$ reaches 0.690. And yet the eigenspace dimensions stay 3 and 1 at every q ; only the eigenvalues q and $-1/q$ move (Section 6). Sixth, a collision inside this corpus's own vocabulary. Paper 153's coproduct is a categorical colimit and this paper's is the structure map Δ — one word, and not even the same kind of thing (Section 7).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED — Hopf algebras, group algebras, the coproduct on an enveloping algebra, the antipode, and the six-vertex R matrix with the Yang–Baxter equation are all standard. No quantum group is constructed here (only a q -deformed R matrix is used). On Paper 152: a mirror, but not the same. That paper set one character table against two groups, and its separator was the Frobenius–Schur indicator, a finer invariant. This one sets one algebra against two coproducts, and its separator is not an invariant but added structure — the difference was outside the algebra from the start, and no finer look would find it. What is added is naming the separator Δ , computing exactly how the fusion rules change over one algebra, showing that grouplike versus primitive is the difference between multiplying and adding, measuring that away from $q = 1$ only the eigenvalues move and the dimensions do not, and separating the two things this corpus calls a coproduct.

1. Notation and position

An algebra A carrying a coproduct $\Delta : A \rightarrow A \otimes A$, a counit $\varepsilon : A \rightarrow \mathbb{C}$ and an antipode $S : A \rightarrow A$ satisfying the usual axioms is a Hopf algebra. Only group algebras of finite groups and a piece of the enveloping algebra of $\mathfrak{sl}_2$ are computed with here. Every number was computed for this paper, and everything that should vanish exactly came out as 0.

Position: Paper 125 separated the name Hopf into three people and remarked in passing that Heinz Hopf left his name on both a fibration and an algebra, which are unrelated objects. This paper opens the algebra.

2. As algebras, they are the same object

Observation 2.1

The groups $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ are both abelian of order four and are not isomorphic: the first has an element of order four, and in the second every element other than the identity squares to the identity.

Verification 2.2: Build each group algebra as a four-dimensional algebra and inspect the structure constants.

group algebra	failure of commutativity	determinant of the character table
$\mathbb{C}[\mathbb{Z}/4]$	0	16
$\mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2]$	0	16

The characters were exactly multiplicative in both cases, the largest discrepancy being 0. (QED)

Observation 2.3

A nonzero determinant makes the characters a basis, and they diagonalise the regular representation. So both group algebras are $\mathbb{C}^4$ — the same object as algebras, with nothing in the multiplication to tell $\mathbb{Z}/4$ from $\mathbb{Z}/2 \times \mathbb{Z}/2$.

3. The core: and yet the tensor products differ

Observation 3.1

A one-dimensional representation is a character, and tensoring two of them multiplies them pointwise. **But that operation is not the multiplication; it uses the coproduct $\Delta(g) = g \otimes g$ — it is Δ that decides that g acts on both systems at once.**

Verification 3.2: The fusion rules. Entry (i, j) names the character that $\chi_i \otimes \chi_j$ is.

$\mathbb{Z}/4$	χ_0	χ_1	χ_2	χ_3
χ_0	0	1	2	3
χ_1	1	2	3	0
χ_2	2	3	0	1
χ_3	3	0	1	2

$\mathbb{Z}/2 \times \mathbb{Z}/2$	χ_0	χ_1	χ_2	χ_3
χ_0	0	1	2	3
χ_1	1	0	3	2
χ_2	2	3	0	1
χ_3	3	2	1	0

(QED)

Verification 3.3: The order under tensoring, that is the least k with $\chi^{\otimes k}$ trivial.

group	orders of the four characters
$\mathbb{Z}/4$	1, 4, 2, 4
$\mathbb{Z}/2 \times \mathbb{Z}/2$	1, 2, 2, 2

(QED)

Observation 3.4

In $\mathbb{Z}/4$ there is a character of order four — tensoring it with itself twice still does not give the trivial representation. In $\mathbb{Z}/2 \times \mathbb{Z}/2$ every character squares to the trivial one. One algebra, two tensor products. What the multiplication left undecided, the coproduct decides.

Observation 3.5

Set this beside Paper 152. There it was one character table and two groups, and the separator was the Frobenius–Schur indicator, a finer invariant — a difference that had been there unseen, outside the table. Here it is one algebra and two coproducts, and the separator is not an invariant but added structure — the difference lies outside the algebra from the start, and looking more finely would never find it.

4. This is why spins add and phases multiply

Observation 4.1

There are two standard shapes of coproduct: the **grouplike** $\Delta(g) = g \otimes g$ and the **primitive** $\Delta(x) = x \otimes 1 + 1 \otimes x$. Under the first, characters multiply; under the second, generators add.

Verification 4.2: Combine two spin-half systems through the primitive coproduct and take the eigenvalues of the total angular momentum squared.

quantity	value	matching $j(j+1)$
eigenvalues of J^2	0, 2, 2, 2	0 for $j = 0$, 2 for $j = 1$
largest $ J_x, J_y - iJ_z $	0	the relations survive

So $j = 1$ three times and $j = 0$ once. (QED)

Observation 4.3

The relations surviving is not an accident. Δ has to be a map of algebras, and that is exactly why the same symmetry still acts on the pair.

Verification 4.4: The fence. The map $\Delta(x) = x \otimes 1$ is also an algebra map and is coassociative. But it breaks the counit axiom $(\varepsilon \otimes \text{id})\Delta = \text{id}$ — a primitive element has $\varepsilon(x) = 0$, so the left side is 0 and never x . The largest entry of x is 0.5, so the failure is total rather than a small correction. (QED)

Observation 4.5

A coproduct is not any way at all of placing two things side by side. Only those ways that satisfy the counit and coassociativity axioms are coproducts.

5. The antipode: where conjugate representations come from

Verification 5.1: In a group algebra put $S(g) = g^{-1}$. The antipode axiom $m(S \otimes \text{id})\Delta(g) = \varepsilon(g)1$ held for every element of both $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$. In the enveloping algebra $S(x) = -x$, and $-x + x = 0 = \varepsilon(x)1$ holds identically. (QED)

Observation 5.2

The antipode is what sends a representation to its dual. Without it there are no conjugate representations and no notion of an opposite charge. Multiplication, coproduct and antipode carry one system, a second system and an opposite system respectively.

6. The fence: cocommutativity, and the dimensions that will not move

Observation 6.1

The coproducts of group algebras and enveloping algebras are **cocommutative**. Swapping the two factors leaves Δ alone, so $V \otimes W$ and $W \otimes V$ are identified by the plain flip, whose square is the identity (measured difference from I : 0).

Verification 6.2: For the non-cocommutative case take the six-vertex R matrix. First confirm that $\hat{R}$ satisfies the Yang–Baxter equation, then measure its square.

q	Yang–Baxter residual	largest $ \hat{R}^2 - I $
1.0	0	0
1.1	2.8×10^{-17}	0.210
1.3	0	0.690
2.0	0	3.000
0.7	5.6×10^{-17}	0.729

(QED)

Observation 6.3

At $q = 1$ braiding is the flip and returns to the identity when squared. Away from $q = 1$ it does not — exchanging two systems stops being an involution. “Exchange” is one word for two operations as well: permuting and braiding.

Verification 6.4: Something does not move even so. The matrix $\hat{R}$ satisfies $(\hat{R} - q)(\hat{R} + q^{-1}) = 0$, and the eigenspace dimensions do not depend on q .

q	Hecke residual	multiplicity of q	multiplicity of $-1/q$
1.0	0	3	1
1.3	5.5×10^{-17}	3	1
2.0	0	3	1
0.7	4.1×10^{-17}	3	1
5.0	8.9×10^{-16}	3	1

(QED)

Observation 6.5

The three-dimensional symmetric piece and the one-dimensional antisymmetric piece stay **3** and **1** however q moves. What moves is only the pair of eigenvalues q and $-1/q$, which become $+1$ and -1 at $q = 1$. The decomposition deforms continuously while the dimensions stay whole numbers — the same shape as the place where Paper 173 wrote that the fence is integrality.

7. The census: this corpus has two coproducts

Observation 7.1

Paper 153 computed categorical coproducts — in **Ab** the coproduct of $\mathbb{Z}/3$ with itself has nine elements, and in **Grp** the coproduct of $\mathbb{Z}/2$ with itself is the infinite dihedral group. **The coproduct here is the structure map Δ .**

	categorical coproduct	Hopf coproduct
what it is	an object (a colimit)	a map
what fixes it	the category	added structure
how many there can be	one up to isomorphism	many on one algebra

Observation 7.2

One word, and not even the same kind of thing. One is settled by the category; the other is placed on the algebra from outside.

8. Conclusion

1. The two group algebras are the same as algebras (Ver. 2.2): both are $\mathbb{C}^4$, both determinants 16.
2. And yet the tensor products differ (Ver. 3.3): character orders 1, 4, 2, 4 against 1, 2, 2, 2.
3. The primitive coproduct adds spins (Ver. 4.2): J^2 has eigenvalues 0, 2, 2, 2.
4. A coproduct is not any placing side by side (Ver. 4.4): $x \otimes 1$ breaks the counit axiom completely.
5. The antipode gives conjugate representations (Ver. 5.1).
6. Away from $q = 1$, exchange is not an involution (Ver. 6.2): 0.690 at $q = 1.3$.
7. And the dimensions stay 3 and 1 (Ver. 6.4).

Closing: Knowing the symmetry group does not settle how two systems are put together. The multiplication says how it acts on one, the coproduct says how to build a second, the antipode says how to build an opposite — three separate pieces of information. Put a different coproduct on the same multiplication and the same symmetry gives different physics.

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