

Incompatibility Begins with the Third Option

— All the Desiderata Hold When There Are Two Candidates: the Separator Is What You Give Up — [Paper 184]

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— *From thinking to syncing.* —

ABSTRACT

This corpus already carries several impossibilities (Papers 25, 163, 178, 179). All of them have the form “this object cannot be built”. This paper adds another form — a set of reasonable-looking requirements that cannot hold at the same time. **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, with two options there is no tension at all. Majority rule satisfied anonymity, neutrality and monotonicity across every profile for two through five voters (Section 2). Second, with three options majority rule already breaks. Cyclic profiles are $1/18$ of all profiles at three voters. Yet at four voters there are none at all, and at six they return — and a counting argument says why (Section 3). Third, and this is the core. Independence collapses the space of rules to a finite one, so the smallest case can be swept by machine. Of 4096 rules, exactly two satisfy transitivity and unanimity together, and both are dictatorships (Section 4). Fourth, there are four escapes and each surrenders something. Give up independence and the Borda count becomes available, but in a four-voter example the social verdict reversed while no voter changed how they ranked a against b (Section 5). Fifth, the fence is the number of options. With two everything holds; with three nothing survives but a dictator (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — the results of Condorcet, Arrow, May and Black are all standard. No political or institutional claim of any kind is made — what is treated here is only the mathematics of whether a set of conditions is compatible. On Paper 25: that paper split “unsolvable” three ways with the separator being which tools are allowed. The separator here is which requirement is surrendered — not a matter of adding tools but of removing demands. What is added is verifying Arrow's theorem in the smallest case by exhaustive search rather than quoting it, showing by a counting argument that cycles cannot occur with four voters, searching out a Borda violation of independence and finding that a strict reversal exists at four voters and not at three, and setting the four escapes side by side by what each gives up.

1. Notation and position

The options are ***a***, ***b*** and ***c***, and each voter ranks them. A collection of everyone's rankings is a **profile**. Of a rule returning a social ranking we ask: **unanimity** (if everyone puts ***x*** above ***y*** then so does society), **independence** (the social ranking of ***x*** against ***y*** depends only on how each voter ranks ***x*** against ***y***), **transitivity** (the social ranking has no cycle) and **non-dictatorship** (no single voter's ranking is always the social one). Every count below is exhaustive.

Position: Paper 179 split “cannot be constructed” three ways, and its impossibilities were about objects — not writable, not searchable, no algorithm. The impossibility here is about a set of demands — objects abound, and none of them meets every condition.

2. With two options there is no tension

Verification 2.1: Take two options, apply majority rule, and check the conditions on every profile.

voters	profiles	anonymity	neutrality	monotonicity
2	4	holds	holds	holds
3	8	holds	holds	holds
4	16	holds	holds	holds
5	32	holds	holds	holds

(QED)

Observation 2.2

All three hold together at every size. While there are only two options nothing has to be given up — majority rule simply works.

3. With three options majority rule already breaks

Verification 3.1: With three options, count the profiles whose pairwise majorities form a cycle.

voters	profiles	cyclic	share
3	216	12	1/18
4	1296	0	0
5	7776	540	5/72
6	46656	180	5/1296
7	279936	21000	875/11664

(QED)

Observation 3.2

Not one at four voters. This is not “rare because small”: it cannot happen. The reason is a count — whichever of the six rankings a voter holds, their contributions to the three margins (a against b , b against c , c against a) sum to either $+1$ or -1 . So over n voters the three margins sum to at most n in size. A cycle needs all three margins to share a sign, and when n is even every margin is even and so at least 2 in size, forcing the sum to be at least 6. At $n = 4$ that cannot be met.

Verification 3.3: The contributions of each ranking.

ranking	three contributions	sum
abc	$(+1, +1, -1)$	$+1$
acb	$(+1, -1, -1)$	-1
bac	$(-1, +1, -1)$	-1
bca	$(-1, +1, +1)$	$+1$
cab	$(+1, -1, +1)$	$+1$
cba	$(-1, -1, +1)$	-1

Every sum is ± 1 . (QED)

Observation 3.4

At six voters they return (180 profiles). The sum can reach 6 only if every voter contributes +1 and all three margins are exactly 2 — it passes by the narrowest margin available.

4. The core: sweeping the smallest case by machine

Observation 4.1

Independence collapses the space. If the social ranking of x against y depends only on each voter's ranking of x against y , then a rule is nothing but one function per pair. With two voters each such function has $2^{2^2} = 16$ possibilities, so there are $16^3 = 4096$ rules in all.

Verification 4.2: Sweep those **4096** and count those satisfying transitivity on every profile together with unanimity.

	count
rules swept	4096
transitive and unanimous	2
of those, dictatorships	2
non-dictatorial survivors	0

(QED)

Observation 4.3

Every survivor is a dictatorship. This is not Arrow's theorem quoted but the smallest case confirmed by machine. “The four conditions are incompatible” is here the result of counting to 4096.

Observation 4.4

An exhaustive sweep is not a proof of the theorem — what was checked is the case of two voters and three options. The general statement is Arrow's (1951).

5. There are four escapes, and each surrenders something

Verification 5.1: Give up independence — the Borda count. Search exhaustively over profiles in which every voter ranks a against b the same way, and see how far the social verdict can move.

voters	verdict can change	strict reversal
2	yes	yes
3	yes	no
4	yes	yes

(QED)

Verification 5.2: The reversal at four voters.

profile	scores	verdict
$abc \mid abc \mid bac \mid bca$	$a = 5, b = 6, c = 1$	b wins
$abc \mid acb \mid bac \mid bac$	$a = 6, b = 5, c = 1$	a wins

No voter changed how they rank a against b — only where c sits moved. (QED)

Observation 5.3

Nobody changed anything about a and b , and the social verdict on them swapped. That is what independence forbids and what the Borda count costs. And at three voters no strict reversal occurs — a win becomes a tie and no further. As with the cycles, the parity of the number of voters is doing work here.

Observation 5.4

Give up transitivity — keep majority rule and live with the cycles (Section 3).

Verification 5.5: Give up the universal domain — place the options on a line $a < b < c$ and admit only rankings with a single peak. The admissible rankings are *abc*, *bac*, *bca* and *cba*.

voters	single-peaked profiles	cyclic
3	64	0
5	1024	0

(QED)

Observation 5.6

The cycles vanish. The price is restricting in advance what anyone is allowed to think — the two rankings *acb* and *cab* are refused at the door.

Verification 5.7: Give up unanimity — the constant rule that always returns $a > b > c$. It is independent (it never looks at the voters), transitive, and has no dictator. Profiles where everyone agrees and the rule contradicts them: **27**. (QED)

Observation 5.8

The fourth escape genuinely exists — and nobody takes it. The list of conditions is long precisely in order to shut escapes of this kind.

6. The census

Observation 6.1

what is given up	what is gained	the price
independence	the Borda count	where an irrelevant third sits flips the verdict
transitivity	majority rule	the social ranking cycles
universal domain	single peaks, median voter	some opinions are refused in advance
unanimity	a constant rule	nobody's opinion counts

Observation 6.2

The fence is the number of options.

options	what happens
2	every condition holds (Ver. 2.1)
3	nothing survives but a dictator (Ver. 4.2)

Observation 6.3

It joins the corpus's row of fences — linearity in Paper 171, integrality in Paper 173, compactness in Paper 177, and here there being only two options. Each stands at a place where adding one more breaks everything.

Observation 6.4

A word on the difference from Paper 25. Its separator was which tools are allowed, and allowing more tools made things solvable. The separator here is which condition is surrendered, and there is nothing to add — only something to remove.

7. Conclusion

1. Two options bring no tension (Ver. 2.1): all three conditions at every size.
2. Three options bring cycles (Ver. 3.1): 1/18 at three voters.

3. Cycles cannot occur at four voters (Obs. 3.2): because the contributions sum to ± 1 .
4. Two survivors out of 4096, both dictators (Ver. 4.2).
5. Borda breaks independence (Ver. 5.2): moving c alone reverses a against b .
6. Single peaks kill the cycles (Ver. 5.5): zero.
7. The fence is the number of options (Obs. 6.2).

Closing: The question “which way of deciding is right” has no answer meeting every condition. The question with an answer is “what will be given up”. And choosing what to give up is not mathematics — mathematics can say only that something must be.

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