

“Rare” Is Measured on a Logarithmic Scale

— The Central Limit Theorem Is the Wrong Instrument in the Tail, and the Fence Is the Single Big Jump —

[Paper 190]

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— *From thinking to syncing.* —

ABSTRACT

When a probability falls through orders of magnitude, saying how many times larger one thing is than another stops meaning anything. This paper shows, against exact combinatorial values, that the right scale for a rare event is the logarithm divided by n . **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, the central limit theorem is the wrong instrument in the tail. For a fair coin reaching a mean of 0.75, the ratio of the exact probability to the gaussian approximation runs 1.63, 1.42, 1.11, 0.690, 0.271 — crossing 1 and continuing to fall (Section 2). Second, and this is the core. The probability itself collapses to 5.1×10^{-20} , while $-\log(\text{probability})/n$ walks quietly to a single number: 0.1939, 0.1508, 0.1346 — with the gap roughly halving each time n doubles (Section 3). Third, the number it walks to is a relative entropy. Changing the target to 3/5, 2/3, 3/4 and 9/10, the value at $n = 800$ matches the relative entropy to within 0.004 (Section 4). Fourth, the fence is the single big jump. Conditioning on a rare large sum, the share carried by the largest term does not move for the exponential law, going from 0.0519 to 0.0513, while for a Pareto law it leaps from 0.1404 to **0.9180** (Section 5). Fifth, one collision of words is recorded. The phrase appearing in this corpus's Paper 42 means the largest discrepancy in a numerical check and has nothing to do with the theory named here (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — Cramér's theorem, rate functions, Sanov's theorem and the single-big-jump principle for heavy tails are all standard. And no theorem is proved here — what is shown is a run of numbers at finite n , and the existence of the limit is quoted. On Paper 187: that paper split why the limit is gaussian into four reasons. The question here is what governs where the gaussian does not, a different one. On Paper 137: that paper examined relative entropy as something that is not a distance. Here the same formula appears as an exponent — one tool, two jobs. What is added is showing against exact values that the central limit theorem is the wrong instrument in the tail, measuring that the logarithm divided by n settles, confirming at four targets that it settles on a relative entropy, and placing the fence at the single big jump with a fair conditioning.

1. Notation and position

Toss a fair coin n times and ask for the probability that the fraction of heads reaches a . This is a sum of binomial coefficients and can be computed **exactly**; below it is kept as an exact fraction. Sampled quantities carry their trial counts.

Position: This corpus keeps a working discipline that when a quantity falls exponentially one reports the exponent and not the ratio. This paper shows where that discipline stands as a theorem inside probability.

2. The central limit theorem is the wrong instrument in the tail

Verification 2.1: The probability that the mean reaches **0.75**, exactly and by the gaussian approximation.

n	exact	gaussian	ratio
20	0.020694733	0.012673659	1.6328932
40	0.0011107169	0.00078270113	1.4190817
80	4.2902799×10^{-6}	3.8721082×10^{-6}	1.1079959
160	$8.7622501 \times 10^{-11}$	$1.2698143 \times 10^{-10}$	0.69004185
320	$5.0734915 \times 10^{-20}$	$1.8720487 \times 10^{-19}$	0.27101279

(QED)

Observation 2.2

The ratio does not settle at one. It crosses one and keeps falling. The central limit theorem describes a window of width $1/\sqrt{n}$ around the mean, and a fixed deviation of 0.25 moves further out of that window as n grows. Using it here is not a matter of poor accuracy: it is the wrong instrument.

3. The core: divide the logarithm by n

Verification 3.1: The same setting, measuring $-\log(\text{probability})/n$.

n	$-\log(\text{exact})/n$	relative entropy	gap
20	0.1938938036	0.1308120359	0.0630818
50	0.1757103435	0.1308120359	0.0448983
100	0.150820182	0.1308120359	0.0200081
200	0.1424967644	0.1308120359	0.0116847
400	0.1375070082	0.1308120359	0.00669497
800	0.134589172	0.1308120359	0.00377714

(QED)

Observation 3.2

The probability itself falls from 0.0207 to 5.1×10^{-20} , eighteen orders of magnitude. The logarithm divided by n walks from 0.194 to 0.135, towards a single number, with the gap roughly halving whenever n doubles ($0.0200 \rightarrow 0.0117 \rightarrow 0.0067 \rightarrow 0.0038$).

Observation 3.3

This is where the corpus's working discipline stands as a theorem — when the answer is exponential, report the exponent and not the ratio. Saying that at $n = 320$ the gaussian approximation is off by a factor of 3.7 carries little. Saying that the exponent is 0.1308 is a statement about every n .

4. The number it walks to is a relative entropy

Verification 4.1: Changing the target and comparing the value at $n = 800$ with the relative entropy.

target	$-\log(\text{exact})/n$	relative entropy	gap
3/5	0.02322630033	0.02013551355	0.00309079
2/3	0.06074561434	0.05663301227	0.0041126
3/4	0.134589172	0.1308120359	0.00377714
9/10	0.3717403699	0.3680642072	0.00367616

(QED)

Observation 4.2

The formula Paper 137 examined as something that is not a distance appears here as an exponent. One formula, two jobs — there it measured how far apart two laws are, here it says how fast a probability dies.

Observation 4.3

And the asymmetry becomes natural here. “How likely is a fair coin to look like a three-quarters coin” and “how likely is a three-quarters coin to look fair” are simply different questions.

5. The fence is the single big jump

Observation 5.1

How does a large sum actually come about? Many terms each a little larger, or one term running away?

Verification 5.2: Sums of 100 terms, 400000 trials. The threshold is set for each law so that the event is equally rare, without which the comparison would be meaningless. The table gives the share of the sum carried by the largest term.

law	unconditioned	top 1%	top 0.1%
exponential	0.0519	0.0521	0.0513
lognormal	0.0807	0.1604	0.2425
Pareto (1.5)	0.1404	0.7278	0.9180

(QED)

Observation 5.3

For the exponential law, conditioning on a rare large sum leaves the largest term's share unmoved — the sum grew because many terms were each slightly larger. For the Pareto law the largest single term takes 92% of the sum — one jump did all of it.

Observation 5.4

So the exponential decay of Section 3 is a statement about the first case only. The fence is a finite exponential moment, and when it breaks the very way rarity is measured changes — a power, not a logarithm. The same fence stands here as the one Paper 186 placed at the tail of the error.

6. The census, and a collision of words

Observation 6.1

instrument	where it looks	how it measures	fence
central limit	near the mean	a window of width $1/\sqrt{n}$	useless for a fixed deviation
large deviations	a fixed deviation	the logarithm over n	needs a finite exponential moment
heavy tails	one jump	a power law	the exponential story fails

Observation 6.2

These three are not a staircase of accuracy. They look at different places. It is not that the gaussian approximation needs a correction term: a different quantity is being measured.

Observation 6.3

A collision of words is worth recording. The phrase appearing in this corpus's Paper 42 refers to the largest discrepancy in a numerical check and has nothing to do with the theory named here. An instance of one phrase covering two things, found inside the corpus itself.

7. Conclusion

1. The central limit theorem is the wrong instrument in the tail (Ver. 2.1): the ratio runs from 1.63 to 0.271, crossing one.
2. The logarithm over n settles (Ver. 3.1): the probability falls eighteen orders while the exponent walks to 0.135.
3. The gap halves as n doubles (Obs. 3.2).
4. It settles on a relative entropy (Ver. 4.1): within 0.004 at four targets.
5. The fence is the single big jump (Ver. 5.2): 0.0513 against 0.9180.
6. One phrase covers two things inside this corpus (Obs. 6.3).

Closing: When an event is called extremely rare, the next question is on what scale it was measured. The probability itself falls through orders of magnitude, so speaking in ratios loses the meaning. Divide the logarithm by n and a single number remains, and that number is what rarity is — except in a world where one jump is enough, where the number does not exist at all.

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