

The Four Means Are One Family, and There Is Only One Equality Fence

— The Separator Is What Each Mean Conserves, and What Actually Breaks Is Positivity — [Paper 194]

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— *From thinking to syncing.* —

ABSTRACT

The harmonic, geometric, arithmetic and quadratic means look like four separate tools. This paper checks numerically that they are four points of one family, and then measures what actually separates them. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the four names are four points on one continuum. The power mean M_p at $p = -1, 0, 1, 2$ agrees with the four definitions to every digit (difference zero), and as p moves the value sweeps continuously from the smallest entry to the largest (Section 2). Second, there is one equality fence, not three. The three consecutive gaps close together, and they close keeping their ratio — at $d = 0.001$ the ratio is $1.000000 : 1 : 1.000000$ (Section 3). Third, this is the core of the paper. Near the fence the four are equally spaced. The ratio of the largest gap to the smallest is 1.0000 at $x = (1, 1.001)$ — but it is 5.0500 at $x = (1, 100)$. The order is universal; the spacing is not (Section 4). Fourth, the separator is what each mean conserves. Applying the four means to four conserved quantities gives a diagonal table — each mean preserves exactly one and destroys the other three (Section 5). Fifth, the fence that actually breaks is positivity, not the equality condition. One negative entry reverses the order itself — at $x = (1, 4, -2)$ the harmonic mean 4.0 sits above the arithmetic mean 1.0 (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the monotonicity of power means, the harmonic–geometric–arithmetic–quadratic chain, and the second-order expansion are all standard. No new proof of any inequality is given — what this paper measures is what the known inequality hides. Weighted means are not treated — every number here uses equal weights. No prescription is given for which mean to use in which situation — the single point made here is that once the conserved quantity is named, the mean is already fixed. Relation to Paper 157: 157 took a census of the 2s in this body of work — that was a census of an exponent, and this is a census of the operation of aggregating. Relation to Paper 136: 136 measured that an exponential average is held by its tail — this paper shows that which tail holds it is decided by the sign of p . What is added is showing numerically that the three gaps close together and in the same ratio, measuring that the four are equally spaced near the fence and not far from it, checking that the four-by-four conservation table is diagonal, and exhibiting a single negative entry that reverses the order itself.

1. Notation and Position

For a tuple of positive numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$, define the power mean for $p \neq 0$ by

$$M_p(\mathbf{x}) = \left(\frac{1}{n} \sum_{i=1}^n x_i^p \right)^{1/p}$$

and for $p = 0$ by

$$M_0(\mathbf{x}) = \exp \left(\frac{1}{n} \sum_{i=1}^n \log x_i \right).$$

Every number below is computed here, in multiprecision arithmetic carrying 40 digits.

Position: The single word “average” is used for at least four different operations. The question here is not which one is correct — the question is what must be moved for the choice of mean to move. In the same shape in which Paper 157 took a census of the roots of the number 2, this paper takes a census of the roots of aggregation.

2. The Four Names Are Four Points of One Family

Check 2.1: for $\mathbf{x} = (1, 9)$, compare each mean computed straight from its own definition with M_p at $p = -1, 0, 1, 2$.

Name	p	M_p	From the definition	Difference
Harmonic	-1	1.8	1.8	0
Geometric	0	3.0	3.0	0
Arithmetic	1	5.0	5.0	0
Quadratic	2	6.40312423743285	6.40312423743285	0

All four agree. The four “different means” are one formula with four numbers put into it.

Check 2.2: move p for $x = (1, 4, 9)$.

p	$M_p(x)$	p	$M_p(x)$
-40	1.027845955565169	0.5	4.0
-8	1.14720049901196	1	4.666666666666667
-4	1.31474201393508	2	5.71547606649408
-2	1.67065805863656	4	6.90452705232703
-1	2.20408163265306	8	7.84666188903874
0	3.30192724889463	40	8.75617591382522

The value increases monotonically in p , and the two ends are $\min x = 1$ and $\max x = 9$ — 1.028 at $p = -40$ and 8.756 at $p = 40$.

Observation 2.3

“The mean” is not four things but one continuum. That continuum is stretched between the smallest and the largest entry, and the four names are four stakes driven into it. The larger p is, the more the answer looks toward the large entries; the smaller, the more it looks toward the small ones.

3. There Is One Equality Fence, Not Three

The four means line up as $M_{-1} \leq M_0 \leq M_1 \leq M_2$. There are three inequality signs here, and so there appear to be three equality conditions. Measure them.

Check 3.1: with $\mathbf{x} = (1, 1 + d)$, measure the three consecutive gaps while shrinking d .

d	$M_0 - M_{-1}$	$M_1 - M_0$	$M_2 - M_1$	The three ratios
2	0.232051	0.267949	0.236068	0.866025 : 1 : 0.881018
1	0.0808802	0.0857864	0.0811388	0.942809 : 1 : 0.945824
0.5	0.0247449	0.0252551	0.0247549	0.979796 : 1 : 0.980192
0.2	0.00453602	0.00455488	0.0045361	0.995859 : 1 : 0.995876
0.1	0.0011898	0.00119115	0.0011898	0.998866 : 1 : 0.998867
0.01	1.24377×10^{-5}	1.24379×10^{-5}	1.24377×10^{-5}	0.999988 : 1 : 0.999988
0.001	1.24938×10^{-7}	1.24938×10^{-7}	1.24938×10^{-7}	1.000000 : 1 : 1.000000
10^{-4}	1.24994×10^{-9}	1.24994×10^{-9}	1.24994×10^{-9}	1.000000 : 1 : 1.000000

The three gaps close together. None of them closes first.

Observation 3.2

There is one fence, not three. $M_p = M_q$ with $p \neq q$ happens, for every pair p, q , exactly when all entries are equal, and that is not three separate conditions but one and the same condition. Three inequality signs stand in a row, so three conditions appear to stand behind them — but the number of symbols and the number of conditions do not match.

Observation 3.3 (this negates a reading that was proposed)

The reading that “each of the four means carries its own independent equality condition” is refuted by Check 3.1. That the ratio of the gaps converges to 1 : 1 : 1 means the three gaps are governed by the same quantity. What that quantity is comes out in the next section.

4. Equally Spaced Near the Fence, Not Far from It

Check 4.1: measure the ratio of the largest gap to the smallest as the spread grows.

x	The four means	The three gaps	Largest/smallest
(1, 1.001)	1.0004998, 1.0004999, 1.0005, 1.0005001	all 1.24938×10^{-7}	1.0000
(1, 1.1)	1.047619, 1.0488088, 1.05, 1.0511898	0.0011898, 0.00119115, 0.0011898	1.0011
(1, 2)	1.3333333, 1.4142136, 1.5, 1.5811388	0.0808802, 0.0857864, 0.0811388	1.0607
(1, 9)	1.8, 3.0, 5.0, 6.4031242	1.2, 2.0, 1.40312	1.6667
(1, 100)	1.980198, 10.0, 50.5, 70.714214	8.0198, 40.5, 20.2142	5.0500

Near the fence the four are equally spaced, and the spacing falls apart as one moves away. At (1, 100) the middle gap is five times the outer ones.

Observation 4.2 (why the spacing is equal)

Writing $x_i = \mu(1 + \delta_i)$ with δ of mean 0 and mean square s^2 , to second order

$$M_p \approx \mu \left(1 + \frac{(p-1)s^2}{2} \right).$$

The right-hand side is linear in p . And the four names sit at $p = -1, 0, 1, 2$, which are equally spaced — so the four means are equally spaced too.

Check 4.3: verify that linearity numerically. Take $x = (1, 1.02, 0.99, 1.01)$, so $\mu = 1.005$ and $\text{Var}/(2\mu) = 6.218905473 \times 10^{-5}$.

p	$(M_p - \mu)/(p - 1)$	Ratio to $\text{Var}/(2\mu)$
-3	$6.218982389 \times 10^{-5}$	1.00001237
-2	$6.219277482 \times 10^{-5}$	1.00005982
-1	$6.219398108 \times 10^{-5}$	1.00007922
0	$6.219344229 \times 10^{-5}$	1.00007055
2	$6.218713073 \times 10^{-5}$	0.99996906
3	$6.218135984 \times 10^{-5}$	0.99987627
4	$6.217384769 \times 10^{-5}$	0.99975547

Wherever p is taken, the same number comes out, and that number is the variance divided by twice the mean.

Check 4.4: at $p = 0$ this says $M_1 - M_0 \approx \text{Var}/(2M_1)$. Shrink the spread and take the ratio.

Spread	$M_1 - M_0$	$\text{Var}/(2M_1)$	Ratio
0.5	0.084289625	0.078125	1.078907195
0.2	0.01263755	0.0125	1.011003964
0.1	0.003133447	0.003125	1.002703038
0.03	0.00028131806	0.00028125	1.000242000
0.01	3.125084×10^{-5}	3.125×10^{-5}	1.000026877
0.003	2.8125068×10^{-6}	2.8125×10^{-6}	1.000002419

Observation 4.5

The quantity governing all three gaps at once is the variance. Near the fence the positions of the four means are fixed by the mean and the variance alone, and the shape of the distribution does not enter. Only once the spread grows do the four begin to see the shape — and that is when the equal spacing breaks.

Observation 4.6

Order and spacing must not be confused. The order $M_{-1} \leq M_0 \leq M_1 \leq M_2$ holds however large the spread. The spacing opens to a factor of five at (1, 100). That an inequality holds implies nothing whatever about the four being comparably far apart.

5. The Separator: What Is Conserved Decides Which Mean

If the four are four points of one family, what decides which one to take? Not the input.

Check 5.1: for $\mathbf{x} = (2, 8)$, replace both entries by each mean and see which of four quantities survives.

Substituted value	$\sum 1/x_i$	$\prod x_i$	$\sum x_i$	$\sum x_i^2$
The original $\mathbf{x} = (2, 8)$	0.625	16	10	68
Harmonic 3.2	0.625	10.24	6.4	20.48
Geometric 4	0.5	16	8	32
Arithmetic 5	0.4	25	10	50
Quadratic 5.8309519	0.34299717	34	11.661904	68

The table is diagonal. Each mean preserves exactly one quantity and destroys the other three.

Observation 5.2 (the separator)

What decides which mean is right is not the data but what is not allowed to move. Keep the sum and it is the arithmetic mean; keep the product and it is the geometric; keep the sum of reciprocals and it is the harmonic; keep the sum of squares and it is the quadratic. p does not come out of the measurement — p comes out of a declaration of what is conserved.

Check 5.3: with growth factors 1.5 and 0.5 over two years, the value after two years is 0.75.

Mean used	Value	Predicted after two years
The truth	—	0.75
Geometric	0.866025	0.75
Arithmetic	1.0	1.0 (off by +33.3%)

The quantity to be kept here is the product. The arithmetic mean does not keep it, and so it manufactures value that was never there.

6. The Wrong Mean Is Not Approximately Right

Check 6.1: driving out at v_1 and back at v_2 over the same road, the average speed is the harmonic mean (the quantity to be kept is **the total time**). Measure the error from using the arithmetic mean.

v_1	v_2	True (harmonic)	Arithmetic	Ratio
60	60	60.000000	60.000000	1.0000
60	90	72.000000	75.000000	1.0417
30	90	45.000000	60.000000	1.3333
30	300	54.545455	165.000000	3.0250
30	3000	59.405941	1515.000000	25.5025
30	10^9	59.999998	500000015	8333333.8333

The error is not bounded. Far from returning a nearby value, the ratio can be made as large as one likes.

Observation 6.2 (the correct answer, by contrast, is bounded)

Since $M_{-1}(v_1, v_2) = 2v_1v_2/(v_1 + v_2) = 2v_1/(1 + v_1/v_2)$, the correct answer never exceeds $2v_1$ however large v_2 is made.

v_2	Harmonic mean	$2v_1 - \text{harmonic mean}$
10^3	58.2524271844660194	1.74757
10^6	59.99820005399838	0.00179995
10^{12}	59.9999999982	1.8×10^{-9}
10^{30}	60.0	1.8×10^{-27}

Even if the return leg were instantaneous, the round-trip average cannot reach twice the outbound speed. The arithmetic mean knows nothing of this ceiling — it grows without bound in proportion to v_2 .

Observation 6.3

Using the wrong mean is not a loss of accuracy. It replaces a bounded quantity by an unbounded one. The two are not even the same kind of number.

7. The Fence That Breaks Is Positivity, Not Equality

Section 3 showed there is only one equality fence. Where, then, is the fence in this system that **can** break? The equality condition cannot: it follows from the definitions. What can break is the domain.

Check 7.1 (put in a zero):

x	Harmonic	Geometric	Arithmetic	Quadratic
(1, 4, 0)	undefined	0	1.666666667	2.380476143
(1, 4, 10^{-12})	3.0×10^{-12}	0.0001587401052	1.666666667	2.380476143

The order survives, but the harmonic mean is dragged down by the single smallest entry. One entry of size 10^{-12} is enough to put the harmonic mean at 3×10^{-12} — no matter how many other entries there are.

Check 7.2 (put in a negative):

x	Harmonic	Geometric	Arithmetic	Quadratic
$(1, 4, -2)$	4.0	not real	1.0	2.6457513
$(5, 1, -3)$	3.4615385	not real	1.0	3.4156503
$(2, 3, -1)$	-18.0	not real	1.3333333	2.1602469

The order itself reverses. At $(1, 4, -2)$ the harmonic mean 4.0 sits above the arithmetic mean 1.0, and at $(2, 3, -1)$ the harmonic mean falls to -18 . Which way it breaks is not fixed either.

Observation 7.3 (the address of the fence)

The fence in this system that can break is “the entries are positive”. The equality condition is not a fence — it is a consequence of the definitions, not an assumption. Positivity is an assumption, and removing it does not merely remove the conclusion: it produces the opposite conclusion. In the same shape in which Paper 171 identified the fence around least squares as linearity rather than normality, the fence around the order of the means is positivity rather than the equality condition.

Observation 7.4

This paper does not say “do not take means of negative numbers”. The arithmetic and quadratic means are defined there and nothing about them breaks. What breaks is the order in which the four were lined up — that is, what breaks is not the individual tools but the relation between them.

8. Which End Holds the Answer

Check 8.1: take $x = (1, 1, 1, t)$ and move t in both directions.

t	Harmonic	Geometric	Arithmetic	Quadratic
10^{-6}	3.999988×10^{-6}	0.03162278	0.7500002	0.8660254
10^{-3}	0.003988036	0.1778279	0.75025	0.8660255
0.1	0.3076923	0.5623413	0.775	0.8674676
1	1.0	1.0	1.0	1.0
10	1.290323	1.778279	3.25	5.074446
10^3	1.332889	5.623413	250.75	500.0007
10^6	1.33333288889	31.62278	250000.8	500000.0

Sending t to zero takes only the harmonic mean with it; sending t to infinity takes only the arithmetic and quadratic means. The harmonic mean saturates at $4/3$ — it is 1.33333288889 already at $t = 10^6$.

Observation 8.2

The sign of p decides which end of the sample holds the answer. Negative p looks to the small end, positive p to the large one. Paper 136 measured that an exponential average is held by the tail of rare paths — 4.3% of the paths carrying 95.2% of the average. That is one thermodynamic instance of what happens when p is taken large.

Observation 8.3

So “which mean to use” is also “which end's information to discard”. The line drawn in Paper 136 — that Jensen's inequality uses only the mean of the distribution and throws its shape away — applies here in the same shape.

9. The Census and the Close

What this paper checked:

1. The four means are M_p at $p = -1, 0, 1, 2$ (Check 2.1, difference zero). They are four points of one continuum whose ends are the smallest and the largest entry (Check 2.2).

2. **There is one equality fence, not three** (Check 3.1): **the three gaps close together, keeping the ratio 1 : 1 : 1.**
3. **Equally spaced near the fence, not far from it** (Check 4.1): **1.0000 against 5.0500. The reason is that to second order M_p is linear in p** (Check 4.3).
4. **The separator is what is conserved** (Check 5.1): **the four-by-four table is diagonal.**
5. **Taking the wrong one is not an approximation** (Check 6.1): **the ratio opens to 8333333.83, while the correct answer saturates at $2v_1$** (Observation 6.2).
6. **The fence that breaks is positivity** (Check 7.2): **one negative entry reverses the order.**
7. **The sign of p decides which end holds the answer** (Check 8.1).

Close: When someone says “take the average”, the next question to ask is which quantity is not allowed to move. Answer that one question and p is fixed, and the mean is decided before it is chosen. Leave it unanswered and the answer depends on which of the four was picked up — and, as measured here, that difference does not politely stay inside the range of “about right”. The chain $M_{-1} \leq M_0 \leq M_1 \leq M_2$ hides all of this under four symbols — there are three signs, but only one fence, and that one is the fence that cannot break.

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