

“Stable” Is Six Separate Words, and What Separates Them Is What You Perturb

— A Saddle and a Centre Imply Nothing About Each Other: the Fence around Linearisation Is Hyperbolicity —

[Paper 195]

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— *From thinking to syncing.* —

ABSTRACT

Paper 185 wrote that the single word “stable” points at two things. This paper adds four more and shows numerically that what separates all six is what you perturb. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, staying near and coming back are different. Around a centre the orbit holds its radius to one part in 10^{14} and stays as close as one likes, and it never once gets closer (Section 2). Second, coming back and coming back at a rate are also different. For $\dot{x} = -x^3$ each halving costs 1.50, 6.00, 24.00, 96.00 — four times the last — and the effective decay rate slides from 1.52×10^{-1} at $T = 10$ to 7.25×10^{-6} at $T = 10^6$ (Section 3). Third, this is the core. Attracting and staying near are different. For $\dot{\theta} = 1 - \cos \theta$ every orbit converges to the fixed point, and yet an orbit starting 0.001 away first travels 3.141380 away before returning — the closer the start, the closer the excursion comes to π (Section 4). Fourth, perturbing the state and perturbing the equation are different. Adding ε to a centre, a perturbation of size $\pm 10^{-6}$ decides by sign alone whether the orbit spirals in or out (Section 5). Fifth, the six are mutually independent. A saddle is structurally stable and Lyapunov unstable; a centre is Lyapunov stable and structurally unstable — neither direction of implication holds (Section 6). Sixth, the fence around linearisation is hyperbolicity. $\dot{x} = -x^3$ and $\dot{x} = +x^3$ have the same linearisation (identically zero), and one converges while the other reaches infinity in finite time (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Lyapunov stability, asymptotic stability, exponential stability, structural stability and the Hartman–Grobman theorem are all standard. No test for stability is proposed — this paper only separates and lines up six existing definitions. Infinite-dimensional systems are not treated — every number here comes from a one- or two-dimensional ordinary differential equation. Structural stability is not decided in general — what is shown here are counterexamples, not a classification of which systems are structurally stable. Relation to Paper 185: 185 treated the classification of defects and noted along the way that “stable” points at two things, topological and energetic. This paper adds Lyapunov, asymptotic, exponential and structural to those two and lines all six up under one separator. Relation to Paper 161: 161 took a census of “fixed point” and found it is not one theorem — that was a census of whether a fixed point exists, and this is a census of in what sense that fixed point is stable. What is added is putting a number on the difference between attracting and staying near, by way of an orbit that travels almost π away, measuring the decay rate sliding to zero across six decades, confirming over two thousand perturbations that a saddle and a centre imply nothing about each other, and showing that the fence is hyperbolicity by way of two systems with identical linearisations.

1. Notation and Position

Consider an ordinary differential equation $\dot{y} = f(y)$ and a fixed point y_* of it. Every integration below uses a fourth-order Runge–Kutta scheme written here, and is checked against the closed form wherever one exists.

Position: Paper 185, in the course of classifying defects, recorded that “stable” points at two things — topologically stable, meaning it cannot be removed by continuous deformation, and energetically stable, meaning it sits at a minimum. This paper continues from there. On the dynamical side there are four more, and what separates all six is the single question of what is being perturbed.

2. Staying Near and Coming Back Are Different

Check 2.1: integrate the centre $\dot{x} = y, \dot{y} = -x$ to $T = 200$ and measure the radius.

Initial radius	Largest radius	Smallest radius	Drift from the start
1.0	1.000000000000	1.000000000000	2.82×10^{-14}
0.1	0.100000000000	0.100000000000	1.78×10^{-15}
0.01	0.010000000000	0.010000000000	3.52×10^{-16}

The radius does not move, and the drift is the size of the integrator's rounding.

Observation 2.2

This is Lyapunov stable and not asymptotically stable. Whatever ε is demanded, starting closer than that keeps the orbit inside ε forever — and yet it never once approaches the fixed point. “A disturbance does no harm” and “a disturbance is undone” are separate claims.

3. Coming Back and Coming Back at a Rate Are Also Different

Check 3.1: integrate $\dot{x} = -x^3$ from $x(0) = 1$. The closed form is $x(t) = 1/\sqrt{1+2t}$.

t	Numeric	Closed form	e^{-t} for comparison
1	0.577350269190	0.577350269190	3.679×10^{-1}
10	0.218217890236	0.218217890236	4.540×10^{-5}
100	0.070534561586	0.070534561586	3.720×10^{-44}
1000	0.022355091700	0.022355091700	0 in double precision

Check 3.2: the time to halve is $3/(2x^2)$ in closed form.

Current value	Time to halve	Times the row above
1	1.50	—
0.5	6.00	4
0.25	24.00	4
0.125	96.00	4

Check 3.3: exponential stability requires $|x(t)| \leq C|x(0)|e^{-\alpha t}$ for some fixed $\alpha > 0$. Measure the effective decay rate $-\log x(t)/t$.

T	$x(T)$	Effective decay rate
10	0.218217890235992	1.5223×10^{-1}
100	0.070534561585860	2.6517×10^{-2}
1000	0.022355091700495	3.8007×10^{-3}
10^4	0.007070891041799	4.9518×10^{-4}
10^5	0.002236062387351	6.1030×10^{-5}
10^6	0.000707106604410	7.2543×10^{-6}

Observation 3.4

The rate slides to zero. Whatever $\alpha > 0$ is offered, a large enough T falls below it. So this system is asymptotically stable and not exponentially stable — “it comes back” can be said, and “how fast” cannot.

The numeric and the closed form differ by only 2.3×10^{-16} at $T = 1000$ — the sliding rate is not integration error.

4. Attracting and Staying Near Are Different Too

Consider $\dot{\theta} = 1 - \cos \theta$ on the circle. The right-hand side is positive except at $\theta = 0$, which is the only fixed point. Since θ can only increase, a point starting just to the right of 0 must **travel all the way round** before it returns.

Check 4.1: measure the largest distance from the fixed point, taken along the circle.

Start	Largest distance reached	Distance at the end
+0.1	3.108379	0.000010
+0.01	3.138380	0.000010
+0.001	3.141380	0.000010
−0.1	0.100000	0.009092
−0.01	0.010000	0.005000

(The largest possible distance on the circle is $\pi = 3.141593$.)

Observation 4.2

Every orbit converges to the fixed point, and yet the fixed point is not Lyapunov stable. It fails even the loose demand $\varepsilon = 1$ — because an orbit starting 0.001 away goes 3.14 away. And the closer the start, the larger the excursion, tending to π .

Observation 4.3

Attracting does not imply staying near. An orbit starting on the left stays inside 0.1 from beginning to end — for the same fixed point, the answer depends on which side the approach comes from. The word “every” in the definition of Lyapunov stability is there for exactly this.

5. Perturbing the State and Perturbing the Equation Are Different

Sections 2 to 4 perturbed the initial state. Here the perturbation is applied to the equation itself.

Check 5.1: perturb the centre to $\dot{x} = y + \varepsilon x, \dot{y} = -x + \varepsilon y$. The radius obeys $r(t) = r(0)e^{\varepsilon t}$.

ε	Numeric $r(200)$	$e^{200\varepsilon}$	Fate
-10^{-2}	0.135335283236625	0.135335283236613	spirals in
-10^{-4}	0.980198673306768	0.980198673306755	spirals in
-10^{-6}	0.999800019998671	0.999800019998667	spirals in
0	1.000000000000012	1	closed orbit
$+10^{-6}$	1.000200020001309	1.000200020001333	spirals out
$+10^{-4}$	1.020201340026755	1.020201340026756	spirals out
$+10^{-2}$	7.389056098929894	7.389056098930650	spirals out

The trailing 12 in the $\varepsilon = 0$ row is the integrator's rounding (1.2×10^{-14}); that orbit is closed.

Observation 5.2

It is the sign, not the size, that decides. Even a perturbation of 10^{-6} sends the orbit outward if positive and inward if negative. So a centre is not structurally stable — however small a neighbourhood is taken, it contains systems with a different phase portrait.

Check 5.3: the same happens to the system of Section 3. Taking $\dot{x} = -x^3 + \varepsilon x$, the linearisation at the origin becomes ε , and for $\varepsilon > 0$ new fixed points appear at $\pm\sqrt{\varepsilon}$.

ε	T	$x(T)$ from $x(0) = 10^{-3}$	Fate
-10^{-2}	2000	0.000000000	toward 0
-10^{-4}	20000	0.000134676	toward 0
0	20000	0.000980581	toward 0, but slowly
$+10^{-4}$	20000	0.005962055	toward $\sqrt{\varepsilon} = 0.0100$
$+10^{-2}$	2000	0.100000000	toward $\sqrt{\varepsilon} = 0.1000$

Observation 5.4

Asymptotic stability does not imply structural stability. The origin of $\dot{x} = -x^3$ is asymptotically stable, and yet adding εx with any positive ε , however small, makes the origin unstable.

6. The Six Are Mutually Independent

Check 6.1: add random perturbations of size 10^{-2} to the coefficient matrix of the saddle $\dot{x} = x, \dot{y} = -y$, two thousand times, and measure the eigenvalues.

Result: in all two thousand the eigenvalues stayed real and kept one positive and one negative sign (the largest imaginary part seen was 0). The phase portrait does not change — a saddle is structurally stable.

Check 6.2: for the same saddle, shrink the initial value of x and run to $T = 30$.

$x(0)$	$x(30)$
10^{-3}	1.068647×10^{10}
10^{-6}	1.068647×10^7
10^{-9}	1.068647×10^4

However close the start, the orbit leaves every neighbourhood — a saddle is Lyapunov unstable.

Check 6.3 (the independence table):

System	Lyapunov stable	Asymptotically stable	Structurally stable
Centre $\dot{x} = y, \dot{y} = -x$	yes	no	no
Stable node $\dot{x} = -x, \dot{y} = -y$	yes	yes	yes
Saddle $\dot{x} = x, \dot{y} = -y$	no	no	yes
$\dot{x} = -x^3$	yes	yes	no

Observation 6.4

The saddle and the centre break the implication in both directions. The saddle is structurally stable and Lyapunov unstable; the centre is Lyapunov stable and structurally unstable. So “structurally stable implies stable” and “stable implies structurally stable” are both false. The two are not stronger and weaker versions of one thing; they are different questions.

7. The Fence Is Hyperbolicity

Check 7.1: $\dot{x} = -x^3$ and $\dot{x} = +x^3$ have linearisations at the origin that are **both identically zero**. Measure where the orbits go.

System	Linearisation at 0	Fate from $x(0) = 0.1$
$\dot{x} = -x^3$	0	0.0224 at $t = 1000$, slowly to 0
$\dot{x} = +x^3$	0	reaches infinity at $t = 50$
$\dot{x} = -x$	-1	2.061154×10^{-10} at $t = 20$

Check 7.2: the closed form for $\dot{x} = +x^3$ is $x(t) = x_0 / \sqrt{1 - 2x_0^2 t}$, which for $x_0 = 0.1$ diverges at $t = 50$.

t	Numeric	Closed form
20	0.129099445	0.129099445
40	0.223606798	0.223606798
49	0.707106781	0.707106781
49.9	2.236067978	2.236067977

Observation 7.3 (the address of the fence)

The fence around “the eigenvalues decide the stability” is hyperbolicity — that is, no eigenvalue having zero real part. Inside the fence the Hartman–Grobman theorem ties the linearisation to the nonlinear system locally, and the eigenvalues do decide. Outside it, two systems with the same linearisation part company into convergence and divergence in finite time. Paper 171 identified the fence around least squares as linearity, and Paper 177 identified the fence around arguments about continuity as compactness — here the fence is hyperbolicity.

Observation 7.4

The centre sits precisely on this fence. Its eigenvalues are purely imaginary, so their real parts are zero — which is why the linearisation has no answer, and why in Section 5 the sign of the perturbation decided. Sections 5 and 7 are two appearances of one fact.

8. The Separator: What You Perturb

Observation 8.1

line up the six senses of “stable” by what is perturbed.

Name	What is perturbed	What is asked
Lyapunov stable	the initial state	does it stay near
Asymptotically stable	the initial state	does it come back
Exponentially stable	the initial state	does it come back at a fixed rate
Structurally stable	the vector field itself	is the phase portrait unchanged
Topologically stable (Paper 185)	the configuration, continuously	can it be removed
Energetically stable (Paper 185)	the scale	does it sit at a minimum

Observation 8.2

The six are not a ladder of strength. The first three do form a ladder (exponential implies asymptotic, asymptotic implies Lyapunov), but the fourth stands outside that ladder — as Check 6.3 showed, neither direction of implication holds. And the last two are not words about a dynamical system at all, but about a configuration of a field.

Observation 8.3

So the sentence “this system is stable” by itself conveys nothing. It conveys something only when what was perturbed is attached to it.

9. The Census and the Close

What this paper checked:

1. **Staying near and coming back are different** (Check 2.1): **the centre holds its radius to one part in 10^{14} and never approaches.**
2. **Coming back and the rate are different** (Checks 3.2, 3.3): **each halving costs four times the last, and the effective rate slides from 10^{-1} to 10^{-6} .**
3. **Attracting and staying near are different** (Check 4.1): **starting 0.001 away, the orbit travels 3.141380 away.**
4. **Perturbing the state and perturbing the equation are different** (Check 5.1): **even at 10^{-6} the sign decides the fate.**
5. **The six are mutually independent** (Check 6.3): **the saddle is structurally stable and Lyapunov unstable, and the centre is the reverse.**
6. **The fence is hyperbolicity** (Check 7.1): **two systems with the same linearisation part into convergence and finite-time divergence.**
7. **The separator is what you perturb** (Observation 8.1).

Close: When told that a fixed point is stable, the next question to ask is stable against what. The initial state, the equation itself, the configuration, or the scale. Those four answers are four separate claims, and as measured here they do not imply one another — a saddle is untroubled by perturbing the equation and runs away from the slightest perturbation of the state, and a centre is the other way round. Paper 185 found two roots in the word “stable” — counted again there are six, and what separates the six is only which thing one picks up and shakes.

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