

“Converges” Is Four Separate Claims, and What Separates Them Is What You May Discard

— A Sequence That Converges at No Point While Its Integral Vanishes: a Null Set, a Small Set, or Everything but a Subsequence — [Paper 197]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

When a sequence of functions is said to converge, that is one of at least four separate claims. This paper shows numerically that the four do not follow from one another, and that what separates them is what you are allowed to discard. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, this is the core. There is a sequence that converges at no point at all while its integral vanishes. Running the typewriter sequence of dyadic intervals to $n = 200000$ and testing 20000 points, the number of points at which it converged was 0, that is 0.000000% — while the L^1 norm of the same sequence is 1.5×10^{-5} (Section 2). Second, a subsequence of it does converge almost everywhere. Keeping only the first interval of each block, at $k = 20$ the number of the 20000 points still returning 1 was 0 (Section 3). Third, the failure of uniformity is not the fault of one endpoint. Multiplying the width of the transition band of x^n by n gives 2.197198, converging to $\log 9 = 2.197225$ — the graph does not flatten: it slides to the right keeping its shape (Section 4). Fourth, riding that sliding point, the value never moves. Evaluating x^n at $x = 1 - 1/n$ gives 0.367879257 at $n = 10^6$, sitting on $1/e = 0.367879441$ (Section 4). Fifth, shrinking the set you are allowed to discard has a price. Discarding a length δ at the right end makes x^n converge uniformly, and at $\delta = 10^{-6}$ the n required is 4605168 — cutting δ by a hundred multiplies n by a hundred (Section 5).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Egorov's theorem, Riesz's theorem and the typewriter sequence are all standard. No test for convergence is proposed — this paper only separates and lines up four existing definitions. The fence around dominated convergence is not treated — the fence of an integrable dominating function is already written in Section 5 of Paper 168, and this paper does not enter there. The spike sequence of unit area is not a finding of this paper either — it is the sequence used in Paper 178, and it appears here as one row of a comparison table. Relation to Paper 177: 177 took a census of the order of $\forall$ and $\exists$ for one function and identified the fence as compactness. The object here is a sequence of functions, and the question is not how δ answers ε but what survives the limit — and the fence is different: here it is the size of the set you may discard. What is added is counting over 20000 points a sequence that converges at none of them, confirming at the same points that a subsequence does converge, locating the failure of uniformity in a sliding band by measuring its width, and measuring the price of shrinking the discarded set across six decades.

1. Notation and Position

For a sequence of measurable functions f_n and a function f , four phrases are used. **Uniform convergence** means $\sup |f_n - f| \rightarrow 0$; **pointwise convergence** means $f_n(x) \rightarrow f(x)$ for every x ; **L^1 convergence** means $\int |f_n - f| \rightarrow 0$; **convergence in measure** means $\mu(\{|f_n - f| > \varepsilon\}) \rightarrow 0$ for every $\varepsilon > 0$. Every number below is computed here.

Position: Paper 177 took a census of the roots of “continuous” for one function. The object here is a sequence of functions, and although the same $\forall$ and $\exists$ stand in a row, the question differs — there it was how δ answers ε , and here it is what survives when the limit is taken.

2. The Core: It Converges at No Point, and the Integral Vanishes

Build the typewriter sequence as follows. Write $n = 2^k + j$ with $0 \leq j < 2^k$, and let f_n be the indicator of $[j/2^k, (j+1)/2^k)$. With each increase of k , a window of half the width sweeps across $[0, 1]$ from left to right.

Check 2.1: the L^1 norm and the measure of the support are both 2^{-k} .

n	Block k	$\ f_n\ _1 = \text{measure of support} = 2^{-k}$
1	0	1
8	3	0.125
64	6	0.015625
1024	10	0.0009765625
65536	16	$1.525878906 \times 10^{-5}$

So this sequence converges to 0 both in L^1 and in measure.

Check 2.2: run n from 1 to 200000 and, at each of 20000 points drawn from $[0, 1]$, count how often $f_n(x)$ was 1 and how often it was 0.

Quantity measured	Value
Points where $f_n(x) = 1$ at least 10 times	20000 (100.00%)
Points where $f_n(x) = 0$ at least 10 times	20000 (100.00%)
Fewest ones seen at any point	17
Points at which the sequence converged	0 (0.000000%)

Observation 2.3

At every point, $f_n(x)$ takes the values 1 and 0 infinitely often. The upper limit is 1 and the lower limit is 0, so not one point converges. And still the integral has fallen to 1.5×10^{-5} . “It converges” and “it converges” are simultaneously true and false of the same sequence — until the sense is named, the sentence has no truth value.

3. And Yet a Subsequence Converges Almost Everywhere

Check 3.1: keep only the first interval of each block, that is $n_k = 2^k$. That subsequence is the indicator of $[0, 2^{-k})$.

k	$n_k = 2^k$	Of the 20000 points, those returning 1
1	2	9933 (49.67%)
4	16	1261 (6.31%)
10	1024	22 (0.11%)
20	1048576	0 (0.0000%)

Observation 3.2

Convergence in measure did not abandon almost-everywhere convergence — it handed it to a subsequence. That is what Riesz's theorem says, and these numbers only build one instance of it. Not any subsequence will do — since the original sequence converges nowhere, only a well-chosen subsequence converges almost everywhere.

4. The Failure of Uniformity Is Not the Fault of One Endpoint

On $[0, 1]$, x^n converges pointwise (0 for $x < 1$ and 1 at $x = 1$), while the supremum is 1 for every n . It is often said that “the single point $x = 1$ is in the way”. That is not right.

Check 4.1: the point where $x^n = t$ is $x = t^{1/n}$. Call the stretch between $t = 0.9$ and $t = 0.1$ the transition band, and multiply its width by n .

n	x with $x^n = 0.9$	x with $x^n = 0.1$	Width $\times n$
10	0.989519258206	0.794328234724	1.951910235
100	0.998946949690	0.977237220956	2.170972873
1000	0.999894645035	0.997700063823	2.194581212
10000	0.999989464004	0.999769767998	2.196960058
100000	0.999998946395	0.999976974414	2.197198124

The width times n converges to $\log 9 = 2.197224577$.

Observation 4.2

The graph does not flatten: it slides to the right keeping its shape. The band narrows like $1/n$, and inside it the value still falls from 0.9 to 0.1. What the failure of uniformity inhabits is not the point $x = 1$ but a band running toward 1.

Check 4.3: ride that band and measure. Take $x = 1 - 1/n$.

n	$x = 1 - 1/n$	x^n
10	0.9000000000000000	0.348678440100000
1000	0.9990000000000000	0.367695424770964
10^5	0.9999900000000000	0.367877601768247
10^6	0.9999990000000000	0.367879257221066

($1/e = 0.367879441171442$.)

Observation 4.4

On a point chasing 1 at speed $1/n$, the value stays at $1/e$ forever. Pointwise convergence says that the sequence goes to 0 at every point, but that is said with the point held fixed; let the point move and it does not go to 0. Uniform convergence is precisely the claim that the point may move.

Observation 4.5

The same 0.3679 appears in Paper 178, but there it is the value of a chosen test function — the agreement of the digits means nothing, and the roots are different.

5. Shrinking the Set You May Discard Has a Price

Check 5.1: discarding a length δ at the right end of $[0, 1]$ leaves $\sup x^n = (1 - \delta)^n$ on $[0, 1 - \delta]$, so x^n does converge uniformly there. Measure the n required.

δ	n needed for $\sup < 0.01$	n needed for $\sup < 10^{-6}$
0.5	7	20
0.1	44	132
0.01	459	1375
10^{-3}	4603	13809
10^{-4}	46050	138149
10^{-6}	4605168	13815504

Observation 5.2

Cutting δ by a hundred multiplies the required n by a hundred. And setting $\delta = 0$ makes no n sufficient — on all of $[0, 1]$ the supremum is 1 forever. What Egorov's theorem guarantees is “possible for every $\delta > 0$ ”, not “possible even at $\delta = 0$ ”. The 4605168 in this table lives in the gap between those two sentences.

6. Without Finite Measure, One Implication Falls Away

Check 6.1: on the real line, let f_n be the indicator of $[n, n + 1]$.

n	Value at a fixed x	Integral	Measure of $\{f_n > 1/2\}$
1	0 once $n > x$	1	1
100	the same	1	1
1000	the same	1	1

Observation 6.2

This sequence converges to 0 at every point, and converges neither in L^1 nor in measure. On a space of finite measure this cannot happen — there, almost-everywhere convergence implies convergence in measure (Egorov). So the table of implications changes with the space — the fence is that the measure be finite.

7. The Separator: What You May Discard

Check 7.1 (the table of implications): for four sequences, set out the truth of the four senses.

Sequence	Pointwise	Uniform	L^1	In measure
x^n on $[0, 1]$	yes	no	yes	yes
n on $[0, 1/n]$ (Paper 178)	yes	no	no	yes
Typewriter	no	no	yes	yes
$[n, n + 1]$ on the line	yes	no	no	no

Every one of them converges in some sense and fails in another. The pointwise column carries a “no” and the L^1 column carries a “no” — neither direction of implication holds.

Observation 7.2 (the separator)

line the senses up by what may be discarded.

Name	What may be discarded
Uniform	nothing
Almost uniform (Egorov)	a set of measure δ , for any $\delta > 0$
Almost everywhere	a set of measure zero
In measure	everything but a subsequence (Riesz)
In L^1	the pointwise information entirely

Observation 7.3

What looks like an order of strength is really an order of permission. Each time the permission is loosened by one step, more sequences get through — the typewriter gets through only once the pointwise information is discarded entirely, and the box on the line gets through L^1 no matter what is discarded.

8. The Census and the Close

What this paper checked:

1. It converges at no point and the integral vanishes (Check 2.2): 0 of 20000 points converged, and the L^1 norm is 1.5×10^{-5} .
2. A subsequence converges almost everywhere (Check 3.1): at $k = 20$, 0 points still return 1.
3. The failure of uniformity lives in a sliding band (Check 4.1): width $\times n$ tends to $\log 9 = 2.197225$.
4. Riding that band the value stays at $1/e$ (Check 4.3): 0.367879257 at $n = 10^6$.
5. The price of shrinking the discarded set (Check 5.1): $n = 4605168$ at $\delta = 10^{-6}$.
6. Without finite measure one implication falls away (Check 6.1).
7. The separator is what may be discarded (Observation 7.2).

Close: The sentence “ f_n converges to f ” has no truth value on its own. Of the typewriter sequence it is true in L^1 , true in measure, and true at not one single point. The four senses look like a ladder of strength, but what builds the ladder is not the functions: it is how much one is prepared to throw away — discard nothing and it is uniform convergence, be willing to lose a null set and it is almost everywhere, settle for keeping a subsequence and it is convergence in measure. The sequence is the same sequence throughout, and what moves is the permission.

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Yuuki YAMAGISHI (山岸 勇輝)

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