

“Straight” Does Not Mean “Shortest”, and There Are Two Fences

— A Flat Cylinder Has No Conjugate Point at All, and Still Stops Being Shortest Past Half Its Circumference —

[Paper 198]

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— *From thinking to syncing.* —

ABSTRACT

A geodesic is defined by having zero acceleration, not by being shortest. This paper measures where the two part company and shows that there are two fences. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the sign of the curvature decides whether a conjugate point exists at all. Integrating the Jacobi equation, the first zero appears at **3.141592653592** for $K = +1$, and for $K = 0$ and $K = -1$ it never appears out to length **40** (Section 2). Second, the second variation changes sign at exactly π . For a great-circle arc on the unit sphere the index form is **+0.144934066848226** at $L = 3$ and **-0.057874312329788** at $L = 3.2$, agreeing with the closed form $(\pi^2 - L^2)/(2L)$ to fifteen digits (Section 4). Third, this is the core. A flat cylinder has no conjugate point at all. And it still stops being shortest past half its circumference — the index form is **+0.822467033424** even at $L = 6$, still positive, and the other way round is still shorter (Section 5). Fourth, what separates the two fences is not a number. The sphere's conjugate point and the cylinder's cut point both sit at π times the radius, so the values agree — what separates them is whether a conjugate point exists (Section 6). Fifth, an optical caustic is the same zero of the same equation. A narrow bundle with focusing power K obeys $y'' + Ky = 0$ and its zero sits at $\pi/\sqrt{K}$ — **1.570796326795** at $K = 4$ (Section 8).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED — the Jacobi equation, conjugate points, cut points and Sturm's comparison theorem are all standard. The location of the cut locus is not determined for manifolds in general — only two examples, the sphere and the cylinder, are treated here. No census of the kinds of curvature is taken — the distinct roots of the word “curvature” are the subject of Paper 91, and this paper does not enter there. Nothing is claimed about any real optical system — Section 8 says only that the equation has the same form. Relation to Paper 111: 111 treated how far light determines geometry — this paper treats how one word, geodesic, splits into three claims. What is added is setting the sign of the curvature against the existence of a conjugate point numerically, confirming the sign change of the second variation by quadrature and in closed form, measuring a surface on which minimality fails while no conjugate point exists at all, and confirming by varying the radius that what separates the two fences is existence rather than value.

1. Notation and Position

A geodesic is a curve whose acceleration vanishes in the covariant sense. That is a **differential condition**, and it says nothing about length. Three claims are kept apart below: **(a) straight** (zero acceleration), **(b) locally shortest** (shortest among nearby curves), **(c) globally shortest** (shortest among all curves). Every integration uses a fourth-order Runge–Kutta scheme written here.

Position: Paper 91 took a census of the distinct roots of the word “curvature” and wrote about intrinsic-ness. This paper does not enter there and measures only where the definition and the intuition part company — and shows that they part in two places.

2. The Sign of the Curvature Decides Whether a Conjugate Point Exists

The separation of a family of geodesics obeys the Jacobi equation $J'' + KJ = 0$. Taking $J(0) = 0$ and $J'(0) = 1$, look for the first zero. Where there is a zero there is a **conjugate point**.

Check 2.1:

Curvature	First zero	Closed form
$K = +1$ (sphere)	3.141592653592	$\pi = 3.141592653590$
$K = 0$ (plane)	none out to $t = 40$	$J = t$ (no zero)
$K = -1$ (hyperbolic)	none out to $t = 40$	$J = \sinh t$ (no zero)

Observation 2.2

Only positive curvature focuses a bundle; zero and negative curvature never do. A conjugate point does not arise because the space is curved but because it is curved positively.

3. The More Curvature, the Sooner the Conjugate Point

Check 3.1: measure the first zero at constant curvature and compare with $\pi/\sqrt{K}$.

K	First zero	$\pi/\sqrt{K}$
0.25	6.283185307178	6.283185307180
1	3.141592653592	3.141592653590
2	2.221441469079	2.221441469079
4	1.570796326795	1.570796326795
9	1.047197551196	1.047197551197

Check 3.2: let the curvature vary. With $K(t) = 1 + \sin^2 t$, which moves between 1 and 2, the first zero is **2.371344171431**, falling between $\pi/\sqrt{2} = 2.221441$ and $\pi = 3.141593$.

Observation 3.3

As Sturm's comparison theorem says, bounding the curvature above and below bounds the zero as well. The curvature decides where the focusing happens, and its sign decides whether it happens.

4. The Second Variation Changes Sign at Exactly π

Take a great-circle arc of length L on the unit sphere and vary it in the normal direction by $f(t) = \sin(\pi t/L)$. The second variation of length, the index form, is $\int_0^L (f'^2 - K f^2) dt$ with $K = 1$.

Check 4.1:

L	Quadrature	Closed form $(\pi^2 - L^2)/(2L)$	Verdict
2	1.467401100272340	1.467401100272339	minimising
3	0.144934066848226	0.144934066848226	minimising
π	-0.000000000000000	0.000000000000000	borderline
3.2	-0.057874312329788	-0.057874312329788	not minimising
3.5	-0.340056514130092	-0.340056514130092	not minimising
4	-0.766299449863830	-0.766299449863830	not minimising

Observation 4.2

Past length π there is a shorter curve right next to it. The great-circle arc is no longer even locally shortest. And this π is the same point at which the conjugate point appeared in Section 2 — the conjugate point is where local minimality ends.

5. The Core: A Flat Cylinder Has No Conjugate Point at All

Take a cylinder of radius **1**. Its Gaussian curvature is **identically zero**: a sheet of paper has merely been rolled, not stretched. So the Jacobi equation becomes $J'' = 0$, its solution is $J = t$, and it **never vanishes, however far one goes**. There is **no conjugate point at all**.

Check 5.1: and still, a geodesic winding around the circumference stops being shortest past half a turn.

Arc length L	The other way round	True distance	Verdict
1	5.283185	1.000000	shortest
3	3.283185	3.000000	shortest
π	3.141593	3.141593	shortest, but not uniquely
3.5	2.783185	2.783185	not shortest
5	1.283185	1.283185	not shortest
6	0.283185	0.283185	not shortest

Check 5.2: measure the index form for the same geodesic. With $K = 0$ it is $\int_0^L f'^2 dt$.

L	Index form
3	+1.644934066848
4	+1.233700550136
6	+0.822467033424

Observation 5.3

The index form is positive at every L . That is, this geodesic is locally shortest all the way, and no nearby curve is shorter. And still it is not shortest for $L > \pi$ — the shorter road is not nearby at all: it is the other way round, far away.

Observation 5.4

So there are two fences. The fence around local minimality is the absence of a conjugate point, and the fence around global minimality is being on the near side of the cut point. They are different fences, and the cylinder separates them completely — no conjugate point exists, and a cut point does.

6. What Separates the Two Fences Is Not a Number

The sphere's conjugate point sits at π times the radius, and the cylinder's cut point also sits at π times the radius. Since the numbers agree, the two may look like one thing.

Check 6.1: vary the radius and measure both.

Radius	Sphere: conjugate point	Sphere: πr	Cylinder: conjugate point
0.5	1.570796326795	1.570796326795	none
1	3.141592653592	3.141592653590	none
2	6.283185307178	6.283185307180	none
3	9.424777960761	9.424777960769	none

Observation 6.2

The two π s do not move independently — both are the length of exactly half a circle of radius r . That the values agree means nothing. What separates the two fences is not a value but whether a conjugate point exists — on the sphere both sit at the same place and cannot be separated, and on the cylinder only one exists and they can.

Observation 6.3

So looking only at the sphere, the two fences look like one. The most symmetric example is the one that hides the distinction best.

7. The Cut Point Is Where Uniqueness Ends, Not Minimality

Check 7.1: on the cylinder of radius 1, move the arc length across the cut point.

L	The other way round	Situation
3.000000000	3.283185307	the unique shortest
$\pi = 3.141592654$	3.141592654	a tie: there are two shortest paths
3.300000000	2.983185307	not shortest

Observation 7.2

What is lost first at the cut point is not minimality but uniqueness. Exactly at the cut point the geodesic is still shortest, only now alongside another one. Minimality goes one step later.

8. An Optical Caustic Is the Same Zero of the Same Equation

The transverse displacement y of a narrow bundle of rays obeys $y'' + Ky = 0$ with K the focusing power. A focus is where y returns to zero, and that is **the same zero of the same equation** as the Jacobi one.

Check 8.1:

Focusing power K	First zero	$\pi/\sqrt{K}$
4	1.570796326795	$\pi/2$
1	3.141592653592	π
0.25	6.283185307178	2π

Observation 8.2

A caustic and a conjugate point are two names for one zero. This is a claim that the equation has the same form, and not a claim about any real optical system.

9. The Census and the Close

Set out the three claims and where each of them fails.

Claim	Fails on the sphere	Fails on the cylinder
Straight (zero acceleration)	never	never
Locally shortest (index form positive)	past πr	never
Globally shortest	past πr	past πR

What this paper checked:

1. Only positive curvature makes a conjugate point (Check 2.1): **3.141592653592** at $K = +1$, and none out to length 40 for $K \leq 0$.
2. The more curvature, the sooner (Check 3.1): the first zero matches $\pi/\sqrt{K}$ to twelve digits.
3. The second variation changes sign at π (Check 4.1): quadrature and closed form agree to fifteen digits.
4. The cylinder has no conjugate point and still stops being shortest (Checks 5.1, 5.2): the index form is **+0.822467033424** even at $L = 6$.
5. What separates them is existence, not value (Check 6.1): both sit at π times the radius, so the sphere hides the distinction.
6. At the cut point uniqueness goes first (Check 7.1).
7. A caustic is the same zero of the same equation (Check 8.1).

Close: The sentence “a geodesic is the shortest path” is conditional twice over. A geodesic is defined by having zero acceleration, which says nothing about length — and getting from there to shortest means clearing two fences, and the two are different fences. The first is the absence of a conjugate point, which only matters where the curvature is positive. The second is being on the near side of the cut point, which matters even where the curvature is zero. A flat cylinder clears the first one without ever touching it and trips over the second — nothing is curved, and going straight still takes the long way round.

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