

What Decides Whether You Come Home Is the Exponent in the Denominator

— In Two Dimensions Return Is Certain and the Mean Time Infinite, and That Certainty Cannot Be Seen by Computing — [Paper 199]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
August 2026

— *From thinking to syncing.* —

ABSTRACT

Whether a point wandering at random over a lattice ever returns to where it started is decided by the dimension alone. This paper shows numerically that the dividing line is an exponent standing in a denominator. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the probability of being back at the origin falls like a power, and the power is half the dimension. Fitting $\log p$ against $\log n$ gives slopes of -0.498567 , -0.997134 and -1.495710 , agreeing with $-d/2$ to within 4.3×10^{-3} (Section 2). Second, what decides is the sum of those probabilities. $\sum n^{-d/2}$ diverges exactly when $d \leq 2$, and the boundary sits at $d = 2$, where the exponent is exactly 1 (Section 3). Third, the first-return probabilities were obtained exactly by the renewal equation. In one dimension they agree with the closed form $1 - \binom{2n}{n}/4^n$ with zero difference, so the tool was verified before use (Section 4). Fourth, this is the core. Return is certain and the mean time to return is infinite. In one and two dimensions the truncated mean return time keeps climbing, reaching 19.5 and 28.0 by $2N = 600$, while in three dimensions it settles at 6.95 — and it settles there not because the walk comes back quickly but because most walks never come back at all (Section 6). Fifth, the certainty in two dimensions cannot be seen by computing. Extrapolating the return probability reaches 0.999954 in one dimension and stops at 0.853135 in two — because the approach is logarithmic, which is not a failure of the computation but the speed of a logarithm (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Pólya's recurrence theorem, the renewal equation and the three-dimensional Green's function are all standard. Pólya's theorem is not proved here — this paper only watches numerically where the theorem switches over. General lattices and weighted walks are not treated — every number here is for the simple symmetric walk. The exact value of the three-dimensional return probability is not derived — the 0.340515 here is an extrapolation of our own, not an evaluation of a closed form. Relation to Paper 94: 94 took a census of the distinct roots of the word “dimension” — this paper is not a census but one mechanism. Relation to Paper 190: 190 wrote that “rare” is measured on a logarithmic scale — Section 7 here shows that the same scale also governs the side of certainty. What is added is reading the dimension off a fitted power, verifying the renewal equation against a closed form before using it, measuring the infinite mean return time as a truncation that keeps climbing, and exhibiting the certainty in two dimensions as an extrapolation that fails to arrive.

1. Notation and Position

On the d -dimensional integer lattice, consider a walk that at each step chooses one of the $2d$ neighbours with equal probability. Starting at the origin, let p_{2n} be the probability of being at the origin after $2n$ steps and f_{2n} the probability of returning to the origin for the first time at step $2n$. Odd steps cannot return, so only even ones are considered. Every number below is computed here.

Position: Paper 94 took a census of the distinct roots of the word “dimension”. This paper takes no census — what it treats is one mechanism, and that mechanism reduces to an exponent standing in a denominator.

2. The Return Probability Falls Like a Power, and the Power Is Half the Dimension

Check 2.1: compute p_{2n} exactly. In one dimension it is $\binom{2n}{n}/4^n$; in two the walk splits into two independent one-dimensional walks after a 45-degree turn, so it is the square of that; in three it is a sum of multinomial coefficients.

Steps $2n$	$d = 1$	$d = 2$	$d = 3$
2	5.000000×10^{-1}	2.500000×10^{-1}	1.666667×10^{-1}
10	2.460938×10^{-1}	6.056213×10^{-2}	1.939193×10^{-2}
40	1.253707×10^{-1}	1.571781×10^{-2}	2.560034×10^{-3}
100	7.958924×10^{-2}	6.334447×10^{-3}	6.549233×10^{-4}
200	5.634848×10^{-2}	3.175151×10^{-3}	2.324181×10^{-4}
300	4.602751×10^{-2}	2.118532×10^{-3}	1.266704×10^{-4}

Check 2.2: fit $\log p$ against $\log n$ over $n = 50$ to 150 .

d	Fitted slope	$-d/2$	Difference
1	-0.498567	-0.5	1.43×10^{-3}
2	-0.997134	-1.0	2.87×10^{-3}
3	-1.495710	-1.5	4.29×10^{-3}

Observation 2.3

The exponent standing in the denominator is half the dimension. The residuals are of order 10^{-3} and come from the $1/n$ correction. Without looking at the walk at all, measuring how fast the probability falls reads off the dimension.

3. What Decides Is the Sum

The expected number of returns to the origin is $\sum_n p_{2n}$. If it diverges the walk returns infinitely often; if it converges it returns only finitely often.

Check 3.1: measure the partial sums.

N	$d = 1$	$d = 2$	$d = 3$
10	2.700138092	0.822389343	0.374121551
50	7.038512976	1.316257723	0.450891793
100	10.326044281	1.534527264	0.469902103
150	12.854281840	1.662798990	0.478384820

Dividing the $d = 1$ partial sum by $\sqrt{N}$ gives 1.049548 and the $d = 2$ partial sum by $\log N$ gives 0.331854: they grow like a square root and like a logarithm respectively.

Check 3.2: look at the same thing away from the walk, as $\sum n^{-d/2}$ to 10^6 terms.

d	$d/2$	Sum to 10^6 terms	Verdict
1	0.5	1998.540145	diverges
2	1.0	14.392727	diverges
3	1.5	2.610375	converges
4	2.0	1.644933	converges ($\pi^2/6 = 1.644934$)

Observation 3.3

The boundary is not on the side of the walk but on the side of the exponent. $\sum n^{-s}$ diverges for $s \leq 1$, and $s = d/2$ crosses that line at exactly $d = 2$.

4. Verifying the Tool First: the Renewal Equation

The first-return probabilities f_{2n} can be extracted recursively from the renewal equation $p_n = \sum_{k=1}^n f_k p_{n-k}$ with $p_0 = 1$. Before using it, check it where the answer is known.

Check 4.1: in one dimension the probability of having returned by step $2n$ has the closed form $1 - \binom{2n}{n}/4^n$.

$2n$	Renewal equation	Closed form	Difference
10	0.753906250000	0.753906250000	0
100	0.920410762613	0.920410762613	0
300	0.953972485581	0.953972485581	0
600	0.967440068665	0.967440068665	0

Observation 4.2

The difference is zero. A newly written tool is first pointed at a problem whose answer is known — and only then at the two- and three-dimensional cases, where it is not.

5. In Three Dimensions, Only About One Walk in Three Comes Back

Check 5.1: measure the probability of having returned by step $2n$ in each dimension.

$2n$	$d = 1$	$d = 2$	$d = 3$
20	0.823802948	0.479596360	0.277437017
100	0.920410763	0.583222193	0.311937514
200	0.943651521	0.616962503	0.320280212
400	0.960130698	0.645882211	0.326201345
600	0.967440069	0.660934075	0.328828768

Check 5.2: fitting against $1/\sqrt{n}$ and extrapolating to $n \rightarrow \infty$ gives a three-dimensional return probability of **0.340515178**, and a corresponding Green's function of $1/(1 - q) = 1.516335126$.

Observation 5.3

That 0.340515 is an extrapolation and not a derivation. Evaluating the closed form is not the business of this paper. What can be said is that in three dimensions the probability of return is not 1 but about a third.

6. The Core: Return Is Certain and the Mean Time Is Infinite

Check 6.1: measure the mean return time truncated at $2N$ steps, that is $\sum_{n \leq N} 2n f_{2n}$.

$2N$	$d = 1$	$d = 2$	$d = 3$
20	3.523941	2.411107	1.207289
100	7.958924	7.409578	2.793136
200	11.269696	12.267055	3.981686
400	15.947721	20.569066	5.662609
600	19.535959	27.987728	6.952444

Observation 6.2

In one and two dimensions the value climbs every time the truncation is pushed out. Returning is certain, and the mean time to return is infinite — the tail is too heavy for a mean to exist.

Observation 6.3

The reason the three-dimensional value settles must not be mistaken. It settles not because the walk returns quickly but because most walks never return at all — a walk that never returns never appears in this sum. The same settled number means speed in one case and absence in the other.

7. The Certainty in Two Dimensions Cannot Be Seen by Computing

Check 7.1: extrapolate the return probability, fitting against $1/\sqrt{n}$ in one and three dimensions and against $1/\log n$ in two.

d	Value at $2n = 600$	Extrapolated to $n \rightarrow \infty$
1	0.967440069	0.999954180
2	0.660934075	0.853135363
3	0.328828768	0.340515178

Check 7.2: follow the probability of not having returned yet, in two dimensions.

$2n$	Not yet returned, q	$1/q$	$\log n$
40	0.470734836	2.124	3.00
100	0.416777807	2.399	3.91
200	0.383037497	2.611	4.61
400	0.354117789	2.824	5.30
600	0.339065925	2.949	5.70

Observation 7.3

Return in two dimensions happens with probability 1. That is a theorem and is not in doubt. And still, after 600 steps a third of the walks have not come back, and the extrapolation stops at 0.853. The reciprocal $1/q$ grows slowly alongside $\log n$: q falls to zero at the speed of a logarithm.

Observation 7.4

This is not a failure of the computation. If $q \sim c/\log n$, then cutting q by a factor of ten means raising the number of steps to the tenth power. Paper 190 wrote that “rare” is measured on a logarithmic scale — here the same scale governs the side of certainty. “Happens with probability one” and “visible in a finite observation” are separate claims.

8. The Boundary Is in the Exponent, Not in the Walk

Check 8.1: measure $\sum n^{-s}$ to 10^7 terms.

s	Sum to 10^7 terms	Verdict
0.5	6323.095124	diverges
0.9	40.688610	diverges
0.99	18.066243	diverges
1.0	16.695311	diverges
1.01	15.464140	converges
1.1	8.589186	converges
1.5	2.611743	converges

Observation 8.2

The partial sums at $s = 0.99$ and $s = 1.01$ are 18.07 and 15.46: barely different. And still one goes to infinity and the other stops. The boundary does not show in the values — what shows is only which side of 1 the exponent is on.

Observation 8.3

So saying that return is “harder” in three dimensions is too weak. What happens in three dimensions is not a change of degree but a change of kind: whether a sum converges. And what decides that kind is only the exponent standing in the denominator of $n^{-d/2}$.

9. The Census and the Close

What this paper checked:

1. The power of the return probability is $d/2$ (Check 2.2): slopes -0.498567 , -0.997134 , -1.495710 .
2. What decides is the sum (Check 3.2): the boundary is $d = 2$, where the exponent is 1.
3. The tool was verified first (Check 4.1): zero difference against the one-dimensional closed form.
4. The three-dimensional return probability is **0.340515** (Check 5.2, an extrapolation, not a derivation).
5. Return is certain and the mean time infinite (Check 6.1): **19.5** and **28.0** by $2N = 600$, still climbing.

6. **The three-dimensional value settles because of absence** (Observation 6.3).
7. **The certainty in two dimensions cannot be seen by computing** (Check 7.1): **the extrapolation stops at 0.853.**
8. **The boundary does not show in the values** (Check 8.1): **18.07 against 15.46.**

Close: Whether a point wandering at random gets home does not depend on the terrain, the stride, or luck. It depends on how fast the probability of being back at the origin falls — and that speed is $n^{-d/2}$, decided by the exponent in the denominator and nothing else. At an exponent of 1 or below the sum diverges and return is certain. Above 1 the sum stops and most walks never come back. Two dimensions sits exactly on that line, and there return is certain, the mean waiting time is infinite, and the certainty is invisible to any finite computation. All three are faces of one fact — that the exponent is exactly 1.

References

1. Pólya, G. (1921). *Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz*. Mathematische Annalen, 84.
2. Watson, G. N. (1939). *Three Triple Integrals*. The Quarterly Journal of Mathematics, 10(1).
3. Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*. Volume 1, third edition. New York: Wiley.
4. Spitzer, F. (1964). *Principles of Random Walk*. Princeton: Van Nostrand.
5. Doyle, P. G., Snell, J. L. (1984). *Random Walks and Electric Networks*. Washington: Mathematical Association of America.
6. Yamagishi, Y. (2026). “Dimension” Is Not One Thing [Paper 94]. Zenodo. 10.5281/zenodo.21758581
7. Yamagishi, Y. (2026). “The Curse of Dimensionality” Is Not One Curse [Paper 182]. Zenodo. 10.5281/zenodo.22104170
8. Yamagishi, Y. (2026). “Rare” Is Measured on a Logarithmic Scale [Paper 190]. Zenodo. 10.5281/zenodo.22106182
9. Yamagishi, Y. (2026). “A Small World” Is Three Separate Claims [Paper 192]. Zenodo. 10.5281/zenodo.22106544
10. Yamagishi, Y. (2026). “Converges” Is Four Separate Claims, and What Separates Them Is What You May Discard [Paper 197]. Zenodo. 10.5281/zenodo.22107590
11. Yamagishi, Y. (2026). “Straight” Does Not Mean “Shortest”, and There Are Two Fences [Paper 198]. Zenodo. 10.5281/zenodo.22107954

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)