

“Complete” Is Four Separate Statements, and What Ties Them Into One Is That the Metric Is Positive Definite

— The Same Open Disc Is Incomplete in the Euclidean Metric and Complete in the Hyperbolic One; on the Lorentzian Side One Notion Evaporates, One Splits in Three, and One Turns Around — [Paper 201]

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— *From thinking to syncing.* —

ABSTRACT

The sentence “that space is complete” is one of at least four separate statements. This paper shows numerically that on the Riemannian side the four are tied into one, and that changing the signature unties them. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, on the Riemannian side the four fail together. On the plane with the origin removed, a geodesic reaches the hole at finite length, a Cauchy sequence loses its destination, and a closed bounded set stops being compact — the three do not fail separately (Section 2). Second, this is the core. On the same set, changing the metric reverses the answer. The radius out to r in the open disc has Euclidean length r and stays bounded, while in the hyperbolic metric it reaches 19.11382792 at $r = 0.99999999$ and has no bound at all (Section 3). Third, in the hyperbolic metric the geodesic never arrives. The unit-speed radial geodesic is $r(s) = \tanh(s/2)$, and its gap to the boundary is $1 - r(s) = 2/(e^s + 1)$, which is never zero for any s (Section 4). Fourth, on the Lorentzian side one of the four cannot even be stated. The interval is not positive definite, so it is not a distance, and the phrase “a Cauchy sequence converges” has nothing to attach to (Section 5). Fifth, another one turns around. For timelike paths from $(0, 0)$ to $(2, 0)$ the straight one has proper time 2, and one bent out to $x = 0.3$ has 1.90787840283389 — the geodesic is not the shortest but the longest (Section 5).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED — the Hopf–Rinow theorem, the Poincaré disc, the Rindler wedge and the maximality of timelike geodesics are all standard. Hopf–Rinow is not proved here — this paper only looks, through examples, at what the theorem demands. The singularity theorems are not treated — what geodesic incompleteness follows from is not the question here. No construction is given separating the timelike, null and spacelike cases — that they split is stated, and no separating example is supplied. Relation to Paper 161: 161 took a census of “fixed point” and used a complete metric space as a hypothesis of Banach's theorem — this paper takes a census of that hypothesis itself. Relation to Paper 198: 198 measured the gap between straight and shortest on the Riemannian side — this paper writes what happens to that gap when the sign turns over. What is added is setting out that the four fail together on a punctured plane, measuring by length that the same open disc is complete or incomplete depending on the metric, putting a number on the reversal of the inequality for timelike paths, and noting that a flat but incomplete example is extendible.

1. Notation and Position

“Complete” has at least four readings. (a) **Geodesically complete**: every geodesic extends to all values of its parameter. (b) **Metrically complete**: every Cauchy sequence converges. (c) **Heine–Borel**: closed bounded sets are compact. (d) **Minimising geodesics exist**: any two points are joined by a geodesic of least length. Every number below is computed here, in multiprecision arithmetic carrying 40 digits.

Position: Paper 161 took a census of “fixed point” and, along the way, noted that Banach's theorem assumes a complete metric space. This paper makes that assumption the subject — and finds that it wears four different faces.

2. On the Riemannian Side, the Four Fail Together

Take the plane with the origin removed, with the ordinary Euclidean metric.

Check 2.1: set the three failures side by side.

Reading	What happens here
Geodesically complete	a geodesic running inward from radius r_0 reaches the hole at length r_0
Metrically complete	the points at radius $1/n$ form a Cauchy sequence whose limit was removed
Heine–Borel	“radius at most 1” is closed and bounded, and that sequence has no convergent subsequence

(Confirming the Cauchy property: the gap between radius $1/n$ and radius $1/2n$ is 0.05 at $n = 10$, 0.005 at $n = 100$, 0.0005 at $n = 1000$ and 0.00005 at $n = 10^4$.)

Observation 2.2

The three did not fail separately. One and the same hole knocks all three readings down at once. That is what the Hopf–Rinow theorem says: on a connected Riemannian manifold the four are equivalent.

3. The Core: On the Same Set, Changing the Metric Reverses the Answer

Measure the open unit disc with two metrics. The set is **the same**.

Check 3.1: measure the straight radius from the centre out to r in both. In the Euclidean metric the length is r ; in the Poincaré metric it is $\log \frac{1+r}{1-r}$.

r	Euclidean length	Hyperbolic length
0.9	0.9	2.944438979
0.99	0.99	5.293304825
0.999	0.999	7.600402335
0.9999	0.9999	9.903437551
0.999999	0.999999	14.50865724
0.99999999	0.99999999	19.11382792

Observation 3.2

It is the same path. Measured one way its length never passes 1; measured the other way it has no bound. So the same open disc is incomplete in one metric and complete in the other — completeness is a property of the metric, not of the set.

Observation 3.3

It is not that one has a boundary and the other does not. The boundary is in the same place. What changed is only how much it costs to get there.

4. In the Hyperbolic Metric the Geodesic Never Arrives

Check 4.1: the unit-speed radial geodesic is $r(s) = \tanh(s/2)$. Measure its gap to the boundary.

Arc length s	$r(s)$	$1 - r(s)$
1	0.462117157260009759	0.537883
5	0.986614298151430289	0.0133857
10	0.999909204262595131	9.07957×10^{-5}
30	0.99999999999812848	1.87152×10^{-13}
60	1 to the 40 digits carried	1.7513×10^{-26}

Observation 4.2

The numbers alone cannot show that it never arrives — at $s = 100$ the gap falls outside the 40 digits carried. So the closed form is used: $1 - \tanh(s/2) = 2/(e^s + 1)$, which is not zero for any finite s . The geodesic extends forever — in this metric the disc is geodesically complete.

Observation 4.3

The same caution from Paper 199 applies here — “never arrives” is invisible to any finite computation. What is visible is only a number shrinking exponentially, and that it is not zero is said by the formula.

5. On the Lorentzian Side, One Evaporates and One Turns Around

Change the signature. Two things happen.

Observation 5.1 (one evaporates)

The interval is not positive definite. The separation of two distinct points can be zero or negative, so there is no metric space — and therefore the phrase “a Cauchy sequence converges” has nothing to attach to. One of the four does not become false: it becomes unstatable.

Check 5.2 (one turns around): measure the proper time of timelike paths from $(t, x) = (0, 0)$ to $(2, 0)$. The straight one gives **2**. Compare a path bent out to $x = X$ at $t = 1$.

X	Proper time of the bent path	Shorter than straight?
0	2	equal
0.1	1.98997487421324	shorter
0.3	1.90787840283389	shorter
0.5	1.73205080756888	shorter
0.9	0.871779788708135	shorter
0.99	0.282134719593318	shorter

Observation 5.3

The straight path is the longest. So the fourth Hopf–Rinow reading — that any two points are joined by a geodesic of least length — is structurally false on this side. Paper 198 measured where straight and shortest come apart on the Riemannian side — here they do not merely come apart: the sign turns over.

Observation 5.4 (one splits in three)

Geodesic completeness also stops being one thing — it splits into timelike, null and spacelike. This paper does not construct examples showing those three independent — it states only that they split.

6. Flat, Still Incomplete, and Extendible

Check 6.1: cut the region $x > |t|$, the Rindler wedge, out of flat spacetime. The curvature is **identically zero**. Measure how long an inertial worldline from $(t, x) = (0, 2)$ moving toward the vertex at speed v takes, in proper time, to leave the wedge.

v	γ	Proper time to the edge
0.1	1.005037815	1.809068067
0.3	1.048284837	1.467598771
0.5	1.154700538	1.154700538
0.9	2.294157339	0.458831468
0.99	7.088812050	0.141776241

Observation 6.2

Nothing is curved, and the geodesic still reaches the edge in finite proper time — the wedge is geodesically incomplete. And this is not a singularity: beyond the edge the rest of flat spacetime continues, so the space is extendible.

Observation 6.3

So “incomplete” and “singular” are different words. Incompleteness can be produced by how a region was cut out — cut a piece away and any space, however flat, becomes incomplete. To be singular takes something more: that it cannot be extended. This has the same shape as the flat cylinder of Paper 198 — there too the curvature was innocent, and the culprit was the global shape.

7. The Fence and the Separator

Observation 7.1 (where the four end up)

Reading	Riemann	Lorentz
Geodesically complete	holds	splits in three
Metrically complete	holds	cannot be stated
Heine–Borel	holds	no corresponding theorem
Minimising geodesics exist	holds	turns around: they are maximal

Observation 7.2 (the fence)

The fence around Hopf–Rinow is not curvature, not dimension and not smoothness, but that the metric is positive definite. Inside the fence the four are one; the moment one steps outside, one evaporates, one splits, and one reverses. Paper 195 identified the fence around linearisation as hyperbolicity, and Paper 171 the fence around least squares as linearity — the fence here is the signature.

Observation 7.3 (the separator)

What separates the four is what is allowed to end after something finite — a parameter (geodesic completeness), a sequence of points (metric completeness), a cover (Heine–Borel), or whether a minimum can be attained (minimising geodesics). On the Riemannian side these four say one thing; on the Lorentzian side they do not.

8. The Census and the Close

What this paper checked:

1. On a punctured plane the three fail together (Check 2.1).
2. The same open disc is complete or incomplete according to the metric (Check 3.1): the same path measures 0.99999999 and 19.11382792.
3. In the hyperbolic metric the geodesic never arrives (Check 4.1): $1 - \tanh(s/2) = 2/(e^s + 1)$ is never zero. The formula says so, not the numbers.
4. On the Lorentzian side metric completeness cannot be stated (Observation 5.1).
5. And shortest turns into longest (Check 5.2): straight gives 2, bent gives 1.90787840283389.
6. Flat, incomplete, and extendible (Check 6.1): incomplete is not singular.

7. The fence is the signature (Observation 7.2).

Close: When told that a space is complete and therefore safe, the next questions are complete in which sense and whether one is inside the fence of the theorem that ties the four together. On the Riemannian side there is no need to ask — Hopf–Rinow makes the four into one. One step outside and the bundle comes apart: one loses its definition, one splits in three, and one changes direction. And the quietest lesson is in Section 3 — whether a space is complete was never a property of the space, but of what was laid on it to measure with. The same disc is a dead end or an endless expanse, according to the ruler.

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