

A Connection Is Not Derived from the Metric: It Is Pinned Only Once Two Demands Are Chosen

— Allowing Torsion in Four Dimensions Returns Exactly 24 Degrees of Freedom, and Exactly 4 of Them Leave the Straight Lines Untouched — [Paper 202]

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— *From thinking to syncing.* —

ABSTRACT

The phrase “fix the metric and the connection follows” quietly adds two demands. This paper counts that the two use up the freedom exactly, and measures what comes back when one of them is dropped. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, demanding only that the metric be preserved does not pin it down. In d dimensions there are d^3 unknowns against $d^2(d+1)/2$ equations, and the freedom left is 2, 9, 24, 50 at $d = 2, 3, 4, 5$ — matching the component count of a torsion tensor, $d^2(d-1)/2$, exactly (Section 2). Second, adding “no torsion” uses the freedom up precisely. The unknowns and the rank agree at every dimension and the freedom left is 0 — the two demands are neither short nor spare (Section 3). Third, the one thing so pinned is the Christoffel symbol. Its violation of metric compatibility is 5.55×10^{-17} and its torsion is exactly zero (Section 4). Fourth, this is the core. Same metric, same lengths, different straight. On the flat plane, choosing another admissible connection without touching the metric at all, two “straight” curves from the same point with the same velocity end up 1.186991 apart (Section 5). Fifth, within the admissible family the choices that leave the straight lines alone number exactly $\binom{d}{3}$. That is 0 at $d = 2$, then 1, 4, 10, 20 — in two dimensions every admissible choice other than Levi-Civita moves the straight lines (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED — the fundamental theorem of Riemannian geometry, torsion, Einstein–Cartan theory and Weyl connections are all standard. No verdict is given on which connection to adopt — this paper counts how many options there are and does not enter the question of which one physics should take. No census of the kinds of curvature is taken — that is the subject of Paper 91. Light and conformal structure are not treated — how a Weyl connection relates to conformal structure is touched on in Paper 111, and this paper does not enter there. Relation to Paper 95: 95 separated convention from fact — this paper draws that line across one object: the metric is given, and the connection is chosen. Relation to Papers 198 and 201: 198 measured where straight and shortest come apart, and 201 wrote that changing the signature turns the direction over — this paper writes what lies before both: that “straight” is fixed by a tool other than the metric. What is added is counting the freedom at four dimensions and checking it against the torsion component count, showing that the two demands fill the rank exactly, measuring how far the straight lines move while the metric is untouched, and confirming at five dimensions that the choices leaving them alone number $\binom{d}{3}$.

1. Notation and Position

A connection is the rule for comparing vectors at different points, written in components as Γ^a_{bc} . It is an object independent of the metric g_{ab} . Two demands are treated below. (i) **Metric compatibility**: parallel transport preserves lengths and angles, that is $\nabla_c g_{ab} = 0$. (ii) **No torsion**: $\Gamma^a_{bc} = \Gamma^a_{cb}$. Every number below is computed here; ranks of linear systems are taken by singular value decomposition with a threshold of 10^{-9} .

Position: Paper 95 separated “convention” from “fact”. This paper draws that line across a connection — the metric is what was laid down to measure with, and the connection is one further choice. How much freedom the two demands consume is then looked at as a number.

2. Preserving the Metric Does Not Pin It Down

The condition $\nabla_c g_{ab} = 0$ is linear in Γ . The unknowns are the d^3 components of Γ^a_{bc} , and the equations number $d^2(d+1)/2$, since c is free and ab is symmetric.

Check 2.1: for a randomly drawn symmetric positive definite metric, measure the rank of the coefficient matrix.

d	Unknowns	Equations	Rank	Freedom left	$d^2(d-1)/2$
2	8	6	6	2	2
3	27	18	18	9	9
4	64	40	40	24	24
5	125	75	75	50	50

Observation 2.2

The freedom left matches the component count of a torsion tensor exactly. The demand that lengths be preserved says nothing at all about the torsion. In four dimensions, **24** options remain untouched.

3. The Second Demand Uses the Freedom Up Exactly

Check 3.1: imposing $\Gamma^a_{bc} = \Gamma^a_{cb}$ reduces the unknowns to $d^2(d+1)/2$. Measure the rank against the same equations.

d	Unknowns	Equations	Rank	Freedom left
2	6	6	6	0
3	18	18	18	0
4	40	40	40	0
5	75	75	75	0

Observation 3.2

At every dimension the unknowns and the equations agree in number, and the rank is full. The two demands are neither short nor spare — one fewer and a family would remain, one more and the solution would vanish. The fundamental theorem of Riemannian geometry can say “unique” because of this balance of counts.

4. And the One So Pinned Is the Christoffel Symbol

Check 4.1: in four dimensions, build the Christoffel formula from a random metric and its derivatives, and measure how far it satisfies the two demands.

Quantity measured	Value
Largest $\nabla_c g_{ab}$	5.551×10^{-17}
Largest torsion (antisymmetric part in the lower indices)	0 exactly

The first is floating-point rounding; the second follows from the formula being built symmetric.

5. The Core: Same Metric, Same Lengths, Different Straight

Take the flat plane with the identity as its metric. By the count of Section 2, the metric-preserving connections form a two-dimensional family. Levi-Civita is the one point of it with zero torsion, and the other points **preserve lengths equally well**.

Check 5.1: pick a member of the family that is not Levi-Civita, and advance an autoparallel from the same point with the same velocity over the same interval of the parameter.

Connection	Endpoint
Levi-Civita (the ordinary straight line)	(3.000000, 0.900000)
Another point of the family	(2.427871, 1.940008)
Separation	1.186991

Observation 5.2

The metric was never changed. So every curve has the same length and every pair of points the same distance. What changed is only which curves count as straight. Paper 198 measured where straight and shortest come apart — here they do not merely come apart: the “straight” side was handed in from outside the metric.

6. The Choices That Leave the Straight Lines Alone Number Exactly $\binom{d}{3}$

Only the part of Γ **symmetric in its two lower indices** enters the autoparallel equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$. So the choices that leave “straight” alone are exactly those **whose symmetric part vanishes**.

Check 6.1: measure the dimension of that subspace inside the admissible family.

d	Family dimension	Leaving the lines alone	$\binom{d}{3}$
2	2	0	0
3	9	1	1
4	24	4	4
5	50	10	10
6	90	20	20

Check 6.2: in four dimensions, take one of each and check.

Connection chosen	Size of torsion	Symmetric part	Shift of the endpoint
One whose symmetric part vanishes	0.3846	1.665×10^{-16}	0.000000000
A generic member of the family	0.2500	0.5000	2.223257376

(Both violate metric compatibility only at the level of 10^{-16} , that is, rounding.)

Observation 6.3

There are choices with nonzero torsion that move the straight lines not at all. “There is torsion” and “straight has changed” are separate claims.

Observation 6.4

And in two dimensions that number is **0**. Because $\binom{2}{3} = 0$: every admissible choice other than Levi-Civita moves the straight lines. Only from three dimensions on does it become possible to carry torsion without moving a single straight line. The room for choice itself depends on the dimension.

7. The Separator and the Fence

Observation 7.1 (the separator)

What separates the metric from the connection is what each is a tool for comparing. The metric compares two vectors at the same point — it gives lengths and angles. The connection compares vectors at different points — it gives what counts as parallel. The second does not follow from the first: the 24 of Section 2 is the size of what does not follow.

Observation 7.2 (the fence)

The fence around “the metric determines the connection” is the extra demand that there be no torsion. Inside the fence it is unique, and the moment one steps out, $d^2(d-1)/2$ degrees of freedom return. And this fence is a choice, not a consequence of a theorem — Einstein–Cartan theory stands outside it with the same metric.

Observation 7.3 (dropping the other demand)

Metric compatibility can be dropped instead — and then parallel transport changes lengths, giving a Weyl connection. This paper does not enter there: how a Weyl connection relates to conformal structure belongs to Paper 111.

8. The Census and the Close

What this paper checked:

1. Preserving the metric does not pin it down (Check 2.1): freedom **2, 9, 24, 50**, matching the torsion count.

2. **The two demands use the freedom up exactly** (Check 3.1): **full rank and zero freedom at every dimension.**
3. **The one so pinned is Christoffel** (Check 4.1): **violation 5.55×10^{-17} , torsion exactly zero.**
4. **Same metric, same lengths, different straight** (Check 5.1): **1.186991 apart on the flat plane.**
5. **The quiet choices number $\binom{d}{3}$** (Check 6.1): **0 at $d = 2$.**
6. **Torsion without moving the lines** (Check 6.2): **shift 0.000000000.**
7. **The fence is the extra demand of no torsion** (Observation 7.2).

Close: “Fix the metric and the geometry follows” is half right. Lengths and angles do follow — but which curves are straight does not. Deciding that means laying down one more tool, and in four dimensions there are 24 ways to lay it. Ordinarily that freedom is erased by a single line, “no torsion” — and that line is a choice, not a theorem. Paper 198 showed straight and shortest coming apart on a flat cylinder, and Paper 201 showed the direction turning over with the signature. What this paper shows lies before both — before they come apart, “straight” was something we decided.

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