

The Power in the Eigenvalue Count Is Half the Dimension

— For Four Boxes of Equal Area the Leading Term Is the Same 15915.5 While the Counts Run from 15782 to 15309: the Difference Is the Perimeter — [Paper 204]

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August 2026

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ABSTRACT

Counting the eigenvalues of a vibration from the bottom up, the number grows like a power. This paper measures that the power is half the dimension, and separates numerically what the leading term hears and what it does not. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the power is half the dimension. Counting Dirichlet eigenvalues of a box and fitting the logarithms, the slopes are 0.500390, 1.002513 and 1.516542 (Section 2). Second, the residue in three dimensions is the boundary term. Raising the ceiling from 3×10^3 to 3×10^5 moves the slope through 1.687516, 1.602565, 1.553497, 1.529462, 1.516542 — monotonically toward 1.5 (Section 3). Third, the leading coefficient returns the volume. It is recovered at ratios 0.999916, 0.998424 and 0.987066, and the shortfall is always on the low side (Section 4). Fourth, this is the core. The leading term hears only the area. For four rectangles of equal area the leading term is 15915.5 in every case, while the actual counts are 15782, 15745, 15619 and 15309 — subtracting the perimeter term brings the four ratios into line at 1.000562, 1.000473, 1.000385 and 0.998162 (Section 5). Fifth, this power is the same power as in Paper 199. Fitting the heat trace gives -0.498008 , -0.994768 and -1.506079 , the same $d/2$ as the return probability of a lattice walk (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Weyl's law, its boundary term and the short-time expansion of the heat kernel are all standard. Weyl's law is not proved — this paper only measures the exponent and the coefficient. Whether the shape can be heard is not discussed — the inverse problem of whether another shape shares a spectrum is the subject of Paper 102, and this paper does not enter there. Only boxes are treated — the numbers come from Dirichlet eigenvalues of rectangles and rectangular blocks, and nothing is claimed for general domains. The three-dimensional slope does not quite reach 1.5 — it is 1.516542, and what the residue is is measured in Section 3. Relation to Paper 199: 199 decided whether a lattice walk comes home by the exponent in a denominator — this paper measures that the same exponent appears in the same shape in the eigenvalue count. What is added is fitting the power of the counting function at three dimensions, measuring that the three-dimensional residue closes monotonically as the ceiling rises, recovering the volume from the leading coefficient and noting the direction of the shortfall, and comparing four boxes of equal area at one common level to show that the difference is the perimeter.

1. Notation and Position

On a box with sides $L_1, \dots, L_d$, the eigenvalues of a vibration vanishing on the boundary are $\lambda = \pi^2 \sum (n_i/L_i)^2$ with $n_i \geq 1$. Write $N(\lambda)$ for the number of eigenvalues below λ . Every number below is computed here, and for each box the enumeration is complete up to its ceiling.

Position: Paper 102 treated Kac's question, whether the shape of a drum can be heard — that is an inverse problem, asking whether the geometry can be recovered from the spectrum. This paper does not enter the inverse problem: it measures the forward direction only, namely what comes out when the eigenvalues are counted.

2. The Power Is Half the Dimension

Check 2.1: count $N(\lambda)$ for unit boxes and fit $\log N$ against $\log \lambda$, over the range from one thirtieth of the ceiling up to the ceiling.

d	Ceiling in λ	Fitted slope	$d/2$	Difference
1	4×10^7	0.500390	0.5	3.90×10^{-4}
2	4×10^6	1.002513	1.0	2.51×10^{-3}
3	3×10^5	1.516542	1.5	1.65×10^{-2}

Observation 2.2

Only the three-dimensional case is off by an order of magnitude more. It will not do to write “it did not fit” and leave it — the next section measures what that difference is.

3. The Residue in Three Dimensions Is the Boundary Term

The second term of the counting function is of size $\lambda^{(d-1)/2}$. Relative to the leading term that is $\lambda^{-1/2}$, so it **ought to fade as the ceiling rises**.

Check 3.1: in three dimensions, repeat the same fit at rising ceilings.

Ceiling in λ	Fitted slope	Difference from 1.5
3×10^3	1.687516	0.1875
1×10^4	1.602565	0.1026
3×10^4	1.553497	0.0535
1×10^5	1.529462	0.0295
3×10^5	1.516542	0.0165

Observation 3.2

The difference closes monotonically, and a hundredfold ceiling cut it to about a eleventh. What was in the way was not error but a second term growing one power more slowly. And it shows its face again in Section 5.

4. The Leading Coefficient Returns the Volume

The leading term is $N(\lambda) \approx \omega_d V \lambda^{d/2} / (2\pi)^d$, with ω_d the volume of the unit ball. Solving for V should **read the volume off the count**.

Check 4.1:

Box	True volume	Recovered	Ratio
1.0	1.000000	0.999916	0.999916
1.0×1.7	1.700000	1.697321	0.998424
$1.0 \times 1.3 \times 0.8$	1.040000	1.026549	0.987066

Observation 4.2

All three fall on the low side. Chance error would not agree in direction — that the direction agrees means something is being subtracted.

5. The Core: the Leading Term Hears Only the Area

Check 5.1: take four rectangles of area **exactly 1** and count at **one common** $\lambda = 2 \times 10^5$. The leading term should be the same for all four.

Box	Area	$N(\lambda)$	Leading term	Ratio
1.000×1.000	1	15782	15915.5	0.991612
2.000×0.500	1	15745	15915.5	0.989288
4.000×0.250	1	15619	15915.5	0.981371
8.000×0.125	1	15309	15915.5	0.961893

The leading term is 15915.5 in all four, and the counts differ from 15782 down to 15309, a spread of 3.0%.

Check 5.2: the four perimeters are **4, 5, 8.5** and **16.25**. Subtract the boundary term $\frac{P\sqrt{\lambda}}{4\pi}$ and compare again.

Box	Perimeter	$N(\lambda)$	Leading — boundary	Ratio
1.000×1.000	4.000	15782	15773.1	1.000562
2.000×0.500	5.000	15745	15737.6	1.000473
4.000×0.250	8.500	15619	15613.0	1.000385
8.000×0.125	16.250	15309	15337.2	0.998162

Observation 5.3

The spread of the ratios fell from 3.0% to 0.24%. What separated the four boxes was the perimeter, not the area. And it is the same thing that was spoiling the three-dimensional slope in Section 3 — one term showing its face in two places.

Observation 5.4

So “the geometry can be read off the eigenvalues” comes in stages: the leading term hears the volume and the next term hears the size of the boundary. Whether anything beyond that hears the shape itself is not the question of this paper — that is Kac’s question, treated in Paper 102.

6. This Power Is the Same Power as in Paper 199

Check 6.1: fit the heat trace $Z(t) = \sum e^{-\lambda t}$.

d	Fitted slope	$-d/2$	Difference
1	−0.498008	−0.5	1.99×10^{-3}
2	−0.994768	−1.0	5.23×10^{-3}
3	−1.506079	−1.5	6.08×10^{-3}

Observation 6.2

Three questions return one exponent.

Question	Power	Measured
Return probability of a lattice walk (Paper 199)	$-d/2$	$-0.4986, -0.9971, -1.4957$
The heat trace (this paper, Check 6.1)	$-d/2$	$-0.4980, -0.9948, -1.5061$
The eigenvalue count (this paper, Check 2.1)	$+d/2$	$+0.5004, +1.0025, +1.5165$

Observation 6.3

They look like different questions — one a wandering walk, one heat, one the pitches of a vibration. That the same number comes out is because the three are one object seen from different sides. This paper does not derive that identity — it is standard, and what is shown here is only that the three measurements line up.

7. The Separator and the Fence

Observation 7.1 (the separator)

What separates what can be read is which term of the expansion is being looked at. The leading term reaches the volume, the next reaches the size of the boundary — each further term makes the audible geometry one step finer. And as measured here, the order is from coarse to fine.

Observation 7.2 (the fence)

The fence around “the power of the eigenvalues tells you the dimension” is looking from high enough up. At $\lambda = 3 \times 10^3$ the slope is 1.69, which is not 1.5 by any reading. Getting inside the fence requires enough height for the boundary term to fade — and that height is further away the higher the dimension.

Observation 7.3

So “measure the power and you know the dimension” is, as a measurement, conditional. The same caution was needed on the lattice of Paper 199 — there too the residue was a correction of order one over the step count. In two settings, a correction of the same shape does the same obstructing.

8. The Census and the Close

What this paper measured:

1. The power is half the dimension (Check 2.1): **0.500390, 1.002513, 1.516542.**
2. The three-dimensional residue is the boundary term (Check 3.1): **a hundredfold ceiling cuts it to an eleventh.**
3. The leading coefficient returns the volume (Check 4.1): **always short on the low side.**
4. The leading term hears only the area (Check 5.1): **one 15915.5 against 15782 down to 15309.**
5. The difference is the perimeter (Check 5.2): **spread from 3.0% to 0.24%.**
6. The same exponent as Paper 199 (Check 6.1).
7. The fence is looking from high enough up (Observation 7.2).

Close: From nothing but counting eigenvalues from the bottom up, out comes first the dimension, then the volume, and then the size of the boundary. They come out coarsest first, and the finer a thing is the deeper the term it hides in. Four boxes of equal area cannot be told apart by the leading term — it returns the same 15915.5 for all of them. To tell them apart one must look one term deeper. And how deep one must look to reach the shape itself — that was Kac's question, treated in Paper 102. What this paper measured is where that question begins.

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