

The Shortest Enclosure Is Not Always a Disc

— On a Flat Torus It Switches to a Band at Area $1/\pi$, and at That Threshold a Descending Method Can No Longer Choose — [Paper 205]

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— *From thinking to syncing.* —

ABSTRACT

The shape enclosing a given area with the shortest boundary is, in the plane, a disc. This paper shows that on a flat torus the answer changes partway to a band, and measures numerically that at the threshold a descending method stops working. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the threshold has a closed form. A disc of area A has perimeter $2\sqrt{\pi A}$ and a band has perimeter 2 whatever A is, and the two cross at exactly $A = 1/\pi = 0.3183098862$ (Section 2). Second, the tool was verified first. Feeding a disc and a band into a perimeter-decreasing flow, neither moves — the changes are -2.16×10^{-4} and $+3.33 \times 10^{-15}$ (Section 3). Third, away from the threshold the flow answers correctly. At area 0.15 the best of 12 runs is 1.3811 against a disc at 1.3729, and at area 0.45 it is 2.0189 against a band at 2.0000 (Section 4). Fourth, this is the core. Near the threshold a descending method cannot choose. The gap to be resolved is only 0.0584 at area 0.30 and 0.0670 at 0.34, while the spread of 12 runs is 1.1535 and 0.9378 — the spread is fourteen to twenty times the gap (Section 5). Fifth, there are two switches. Past area $1 - 1/\pi = 0.681690$ it is the hole that becomes a disc — the answer is symmetric about a half (Section 6).

Scope of this paper (scope note): **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED** — the isoperimetric inequality, the isoperimetric regions of a flat torus and threshold dynamics are all standard. It is not proved that the band and the disc are the only candidates — only those two, and the disc-shaped hole, were compared, and nothing shorter is ruled out. The numerical flow does not guarantee a global minimum — Section 5 is precisely a measurement of that absence of guarantee. The perimeter estimator is not exact — after calibration against known shapes a residue of about 0.7% remains on a disc. Relation to Paper 198: 198 showed that on a flat cylinder “straight” takes the long way round — this paper shows that on a flat torus the shortest enclosure stops being a disc. In both, the curvature is innocent and the culprit is the global shape. Relation to Paper 193: 193 wrote that an estimate is wrong precisely where one would want to use it — Section 5 here has the same shape: a descending method cannot choose precisely where the answer switches. What is added is putting the crossing of the two candidates in closed form, verifying the perimeter-decreasing flow on known shapes before using it, confirming that the flow answers correctly on both sides away from the threshold, and measuring over 12 runs each that near the threshold the spread exceeds the gap by more than fourteenfold.

1. Notation and Position

On a flat square torus of side **1**, a square with opposite edges glued, take a region of area **A** and call the length of its boundary the **perimeter**. The perimeter is measured with an **isotropic estimator** — counting grid edges is not used, since it makes a diagonal boundary too long by a factor of $\sqrt{2}$. Every number below is computed here.

Position: Paper 198 showed that on a flat cylinder “straight” takes the long way round — the curvature was identically zero and the culprit was the global shape. This paper asks the same question in terms of area and perimeter: on something flat, from where does the shortest enclosure stop being a disc?

2. The Two Candidates Cross at Exactly $1/\pi$

On a torus there is a candidate the plane does not have: **a band running all the way round**. A band has no ends, so its boundary is only **two cuts**, of total length **2** regardless of the area.

Check 2.1:

Area A	Disc, $2\sqrt{\pi A}$	Band	Shorter
0.100000	1.120998	2.000000	disc
0.200000	1.585331	2.000000	disc
0.300000	1.941626	2.000000	disc
$1/\pi = 0.318310$	2.000000	2.000000	equal
0.350000	2.097196	2.000000	band
0.450000	2.377996	2.000000	band

Observation 2.2

Solving $2\sqrt{\pi A} = 2$ gives $A = 1/\pi$. The threshold is not an approximation but exactly that number. This paper does not prove that these two are the only candidates — only two were compared.

3. Verifying the Tool First

Check 3.1: calibrate the perimeter estimator on shapes whose answer is known. A residue remains even after calibration.

Shape	True perimeter	Estimate after calibration
Disc of area 0.2	1.585331	1.597002
Band	2	1.984925

The residue is about 0.7%, and that 0.7% is the floor on what can be seen below.

Check 3.2: feed a disc and a band into a perimeter-decreasing flow, an area-preserving threshold dynamics. Both are **already candidates for the optimum**, so neither should move.

Shape fed in	Perimeter before	After	Change
Disc of area 0.20	1.597002	1.596787	-2.16×10^{-4}
Disc of area 0.10	1.129082	1.128550	-5.32×10^{-4}
Band of area 0.40	1.984925	1.984925	$+3.33 \times 10^{-15}$
Band of area 0.60	1.984925	1.984925	$+2.22 \times 10^{-16}$

Observation 3.3

Both are fixed points. The flow does not destroy good shapes — and only then is it used where the answer is unknown.

Observation 3.4 (the first attempt failed)

The first run of this paper was under-converged — the diffusion length was too short, the flow stopped before the noise had gathered into one blob, and the final perimeters came out between 2.5 and 3.9, matching neither a disc nor a band. Without the verification placed first, those numbers would have been written down.

4. Away from the Threshold, the Flow Answers Correctly

Check 4.1: run the flow from smooth random initial shapes, **12** times at each area, and take the best.

Area	Disc	Band	Best of the flow	Nearer to
0.1500	1.3729	2.0000	1.3811	disc
0.4500	2.3780	2.0000	2.0189	band

Observation 4.2

On both sides of the threshold the flow agrees with the closed form, to 0.6% at 0.15 and to 0.9% at 0.45.

5. The Core: Near the Threshold a Descending Method Cannot Choose

Check 5.1: repeat near the threshold, and measure the **spread** of the **12** runs as well.

Area	Gap to resolve	Best	Median	Spread
0.1500	0.6271	1.3811	1.9339	1.0008
0.2600	0.1924	1.8503	2.4933	2.5780
0.3000	0.0584	1.9757	2.1500	1.1535
$1/\pi$	0.0000	2.0120	2.2993	0.9454
0.3400	0.0670	2.0161	2.2155	0.9378
0.4500	0.3780	2.0189	2.8598	5.2529

Observation 5.2

At area 0.30 the gap to be resolved is only 0.0584, while the 12 runs scatter over 1.1535 — the spread is twenty times the gap. At 0.34 it is 0.9378 against 0.0670, fourteen times.

Check 5.3: look at the gap itself, finely, around the threshold.

Area	Disc	Band	Gap as a percentage
0.250000	1.772454	2.000000	11.377%
0.290000	1.908991	2.000000	4.550%
0.310000	1.973721	2.000000	1.314%
$1/\pi$	2.000000	2.000000	0%
0.330000	2.036394	2.000000	1.820%
0.350000	2.097196	2.000000	4.860%

Observation 5.4

Just either side of the threshold the two shapes differ by only about one per cent — the same order as the 0.7% residue of the estimator measured in Section 3. So even if the flow worked perfectly, the measuring could not tell them apart.

Observation 5.5

And at the threshold itself the gap is exactly zero. The two shapes have the same length, and a procedure that only decreases the perimeter has no reason to prefer either. “Where the answer switches” is also where descent stops being a method. Paper 193 wrote that an estimate is wrong precisely where one would want to use it — here descent falls silent precisely where one most wants to ask.

6. There Are Two Switches, Symmetric About a Half

Check 6.1: as the area grows, it is **the hole** that becomes a disc. Set the three candidates side by side.

Area	Disc	Band	Hole is a disc	Shortest
0.100000	1.120998	2.000000	3.362995	disc
0.250000	1.772454	2.000000	3.069980	disc
$1/\pi$	2.000000	2.000000	2.926836	threshold
0.500000	2.506628	2.000000	2.506628	band
$1 - 1/\pi$	2.926836	2.000000	2.000000	threshold
0.800000	3.170662	2.000000	1.585331	hole is a disc

Observation 6.2

The switches are at $1/\pi = 0.318310$ and $1 - 1/\pi = 0.681690$, symmetric about a half. At area one half the disc and the disc-shaped hole are exactly equal in length — and both lose to the band.

7. Changing the Sides Moves the Threshold

Check 7.1: keep the area at **1** and change the sides of the torus. The shortest band cuts across **the shorter side** twice, so its perimeter is **$2 \min(a, b)$** .

Sides	Shortest band	Threshold area	Times $1/\pi$
1.00×1.00	2.000000	0.318310	1.0000
0.80×1.25	1.600000	0.203718	0.6400
0.50×2.00	1.000000	0.079577	0.2500
0.25×4.00	0.500000	0.019894	0.0625

Observation 7.2

The threshold is $\min(a, b)^2/\pi$ — the more the torus is squeezed, the sooner the band wins. At 0.25×4.00 the switch comes at barely 2% of the area.

8. The Separator and the Fence

Observation 8.1 (the separator)

The answer does not change because of curvature — the torus is flat. What changes it is whether one can go all the way round, which is the global shape. The same shape as the flat cylinder of Paper 198: the curvature was innocent both times.

Observation 8.2 (the fence)

The fence around “the shortest enclosure is a disc” is being in the plane — more precisely, that no other candidate exists. On a torus a band appears, and that alone changes the answer. While the area is small the plane's answer survives: below $1/\pi$ the disc wins.

Observation 8.3

And the other fence this paper measured lies on the side of the method. Descent is usable only while one answer is clearly better, and it does not work where two are level. That place can be written in one line in closed form — $A = 1/\pi$.

9. The Census and the Close

What this paper checked:

1. The threshold is $1/\pi$ in closed form (Check 2.1).
2. The tool was verified first (Check 3.2): disc and band are fixed points. The first run failed, under-converged (Observation 3.4).
3. Away from the threshold the flow is right (Check 4.1): to 0.6% and 0.9%.
4. Near it, it cannot choose (Check 5.1): the spread is fourteen to twenty times the gap.
5. At the threshold the gap is exactly zero (Check 5.3): descent has no reason to prefer either.
6. Two switches, symmetric about a half (Check 6.1).
7. Squeezing makes the band win sooner (Check 7.1): the threshold is $\min(a, b)^2/\pi$.

Close: “The circle is shortest for a given area” is a statement that closes only in the plane. One road that goes all the way round is enough to create a new candidate, the band, and the moment the area passes $1/\pi$ the band wins. The curvature is still zero, and what changed is only how things are joined up — the same culprit as the flat cylinder of Paper 198, where straight took the long way round. And the other thing measured here is a limit of method: where the answer switches, the two shapes have the same length, and a procedure that only decreases the perimeter loses its reason to choose. The threshold is where descent no longer tells you anything — and what is needed there is not to run it longer but to write one line in closed form.

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