

Curvature Does the Same Work as Dimension

— In Hyperbolic Space Eighty-Six Per Cent of a Ball Lives Forever in Its Outermost Unit Shell: to Imitate That
in Flat Space the Dimension Must Grow Like Twice the Radius — [Paper 208]

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ABSTRACT

How the volume of a ball grows with its radius splits into three behaviours according to the curvature alone. This paper measures the three and shows numerically that the crowding produced by curvature has the same shape as the crowding produced by dimension. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the growth splits three ways. In the three-dimensional models the volume of a ball of radius 4 is 4657.344588 at curvature -1 and 268.0825731 at curvature 0, while at curvature $+1$ it has already finished at radius π (Section 2). Second, in hyperbolic space only the outside grows. The share of the ball lying between $r - 1$ and r is 0.8646647168 at $r = 20$, agreeing with $1 - e^{-2} = 0.864664716763$ to ten digits — however large the radius is made, eighty-six per cent goes on living in the outermost unit shell (Section 4). Third, this is the core. The same crowding can be bought in two different ways. The dimension needed to reproduce that share in flat space is 4.93 at $r = 3$, 18.98 at $r = 10$ and 98.997 at $r = 50$ — the dimension has to keep growing like twice the radius (Section 5). Fourth, the ratio never turns back. Dividing the volume at greater curvature by the volume at lesser, the ratio fell at every one of six radii, checked by an assertion — the content of the comparison theorem is not its value at one radius but that it never reverses (Section 6). Fifth, a hyperbolic ball never becomes mostly interior. The ratio of surface to volume goes to zero as $3/r$ in flat space, and settles at 2.0 by $r = 20$ in hyperbolic space (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the volumes of the constant-curvature models and the Bishop–Gromov comparison theorem are standard. The comparison theorem is not proved — what is checked here is that the ratio between two models falls monotonically, and nothing is claimed about a general manifold with a lower bound on its Ricci curvature. Only three dimensions are measured — Section 5 alone varies the dimension, and what it varies is the dimension of flat space. Nothing is claimed about any real space. Relation to Paper 182: 182 measured the crowding produced by dimension — this paper measures the same crowding produced by curvature. One phenomenon with two separate causes, and Section 5 is the demonstration that either will buy it. Relation to Papers 60 and 63: the corpus already treats hyperbolic geometry in velocity space — that is about where relativistic velocities live, and this is about how volume grows. What is added is obtaining the three growth laws by quadrature and in closed form, measuring at six radii that the outer shell settles on a constant share in hyperbolic space, showing that the dimension needed to reproduce that share in flat space is twice the radius, and confirming the monotonicity of the ratio with an assertion.

1. Notation and Position

Consider the volume of a geodesic ball of radius r in the n -dimensional model of constant curvature K . In three dimensions there are closed forms: $\pi(\sinh 2r - 2r)$ at $K = -1$, $\frac{4}{3}\pi r^3$ at $K = 0$, and $\pi(2r - \sin 2r)$ at $K = +1$ for $r \leq \pi$. The numbers below come from multiprecision quadrature, checked against those forms.

Position: Paper 182 treated the curse of dimensionality and measured that raising the dimension gathers the volume toward the outside. This paper measures the same gathering from the side of curvature — and Section 5 shows that the two can be exchanged.

2. The Growth Splits Three Ways

Check 2.1: measure the volume at several radii in the three-dimensional models.

r	$K = -1$	$K = 0$	$K = +1$
0.5	0.5504107829	0.5235987756	0.4980335895
1	5.110932706	4.188790205	3.426543191
2	73.16743277	33.51032164	14.94393577
3	614.8510174	113.0973355	19.7273656
4	4657.344588	268.0825731	(finished at $r = \pi$)

Observation 2.2

One grows exponentially, one as a cube, and one stops partway. The same phrase, “the volume of a ball”, points at three separate behaviours according to the sign of the curvature alone.

3. Checking the Tool

Check 3.1: set the quadrature against the closed forms.

r	Curvature	Quadrature	Closed form
1	-1	5.11093270570829	5.11093270570829
1	0	4.18879020478639	4.18879020478639
3	-1	614.851017405332	614.851017405332
3	0	113.097335529233	113.097335529233

They agree to fifteen digits. Every number below comes from that quadrature.

4. In Hyperbolic Space Only the Outside Grows

Check 4.1: measure the share of the whole ball taken by the shell between $r - 1$ and r .

r	$K = -1$	$K = 0$	$K = +1$
2	0.930147437	0.875	0.7707067775
3	0.880999737	0.7037037037	0.242476868
5	0.8652687948	0.488	(finished)
8	0.8646673804	0.330078125	(finished)
12	0.8646647182	0.2297453704	(finished)
20	0.8646647168	0.142625	(finished)

$(1 - e^{-2} = 0.864664716763.)$

Observation 4.2

In flat space the shell empties out, from 0.875 down to 0.142625. In hyperbolic space it settles on a constant share, agreeing with $1 - e^{-2}$ to ten digits by $r = 20$. However large the radius is made, more than eighty-six per cent goes on living in the outermost unit shell.

Observation 4.3

The spherical column loses its meaning partway — it is already down to 0.242476868 at $r = 3$, because the ball covers the whole sphere at $r = \pi$.

5. The Core: the Same Crowding, Bought Two Ways

In flat n -dimensional space the outer unit shell takes a share of $1 - (\frac{r-1}{r})^n$. What must n be to match the hyperbolic three-dimensional value of $1 - e^{-2}$?

Check 5.1:

r	Dimension n giving the same share	$2r$
3	4.932606925	6
5	8.962840235	10
10	18.98244316	20
20	38.99145149	40
50	98.99663291	100

Observation 5.2

To produce the same crowding in flat space, the dimension must keep growing like twice the radius. The curvature side was set to -1 once and increased nothing thereafter. One phenomenon, bought by paying in dimension on one side and in curvature on the other.

Observation 5.3

So the observation “the volume is gathered toward the outside” does not by itself say what caused it. Paper 182 measured the gathering caused by dimension and this paper the gathering caused by curvature — the shape that shows is the same, and the roots are separate. To tell which, vary the dimension and see whether the share moves.

6. The Ratio Never Turns Back

Check 6.1: divide the volume at the greater curvature by the volume at the lesser.

r	$K = +1$ over $K = 0$	$K = 0$ over $K = -1$
0.5	0.951174091153	0.951287278359
1	0.818026929881	0.819574517212
1.5	0.635306664876	0.641219749076
2	0.445950233935	0.457995044653
2.5	0.286028365184	0.301045763042
3	0.174428208283	0.183942666317

Every step is smaller than the last. The computation carries an assertion that stops it if a ratio rises, and it never stopped.

Observation 6.2

The content of the comparison theorem is not its value at one radius — it is that it never reverses. An inequality at one point could hold by accident; being monotone could not. What was checked here is only between models, and says nothing about a general manifold with a lower bound on its Ricci curvature.

7. A Hyperbolic Ball Never Becomes Mostly Interior

Check 7.1: divide the surface of the ball by its volume.

r	$K = -1$	$K = 0$
1	3.3957378	3.0
2	2.259195051	1.5
5	2.001635891	0.6
10	2.000000157	0.3
20	2.0	0.15

Observation 7.2

In flat space it goes to zero as $3/r$ — the larger the ball, the more it becomes “interior”. In hyperbolic space it settles on 2 — however large it is made, the proportion of skin to interior does not change. It is the eighty-six per cent of Section 4, said another way.

8. The Separator and the Fence

Observation 8.1 (the separator)

The same observation, “the volume is gathered toward the outside”, has two separate causes — dimension (Paper 182) and curvature (this paper). To separate them, vary the dimension: crowding from dimension goes away when the dimension is lowered, and crowding from curvature stays put in three dimensions.

Observation 8.2 (the fence)

The fence around “the volume of a ball grows like the n -th power of the radius” is that the curvature is zero. Outside the fence the form of the growth itself changes — exponential if negative, and finishing partway if positive. And this is not a difference of degree between faster and slower growth: the $K = +1$ column has nothing at all beyond radius π .

Observation 8.3

In Paper 206 the dimension of a thin cylinder had no answer until a scale was named — here the growth of a ball's volume is not settled until a curvature is named. Both have the same shape: the name of the quantity is not enough.

9. The Census and the Close

What this paper measured:

1. Three kinds of growth (Check 2.1): **exponential, cubic, and finishing partway.**
2. The tool was checked (Check 3.1): **quadrature and closed form agree to fifteen digits.**
3. The hyperbolic shell keeps **86%** (Check 4.1): **ten digits of $1 - e^{-2}$.**
4. Imitating it needs twice the radius in dimensions (Check 5.1): **$n = 98.997$ at $r = 50$.**
5. The ratio never turns back (Check 6.1): **confirmed with an assertion.**
6. Surface over volume settles on **2** (Check 7.1).
7. The fence is that the curvature be zero (Observation 8.2).

Close: To arrange, in flat space, that almost all of a ball's volume lies near its surface, there is no way but to keep raising the dimension — double the radius and the dimension must double to keep up. Set the curvature to -1 , however, and it holds in three dimensions, from beginning to end. The same crowding is sold at two completely different prices. And since the shape that shows is the same, looking at the shape will not tell you which was bought — that is learnt only by moving the dimension and watching.

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