

Equal Volume Is Not Enough to Cut and Reassemble

— In the Plane Equal Area Always Suffices, and in Space a Regular Tetrahedron Cannot Become a Cube: Its Dihedral Angle $\arccos(1/3)$ Is Not a Rational Multiple of a Half Turn — [Paper 209]

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— *From thinking to syncing.* —

ABSTRACT

In the plane, “equal area” and “can be cut up and rearranged into one another” say the same thing. In space they do not. This paper measures the difference and shows that what makes the difference is the dimension. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, of the five regular solids only the cube has a dihedral angle that is a rational multiple of a half turn. The cube gives exactly 0.5; the tetrahedron 0.391826552030607, the octahedron 0.608173447969393, the dodecahedron 0.647583617650433 and the icosahedron 0.76772047280123 (Section 2). Second, that the tetrahedron's angle is not one can be settled with integers alone. Writing $\cos n\theta = a_n/3^n$ turns the problem into the integer recurrence $a_{n+1} = 2a_n - 9a_{n-1}$, and a_n is divisible by three for no n from 1 to 199 — so $\cos n\theta$ is never ± 1 (Section 3). Third, it nevertheless comes arbitrarily close. The distance from $n\theta$ to a whole number of half turns falls to $3.945928121 \times 10^{-6}$ at $n = 38882$ — coming close and arriving are two different things (Section 4). Fourth, this is the core. Scaled to the same volume of 1, the two solids still cannot be carried into one another. The cube has edge 1 and dihedral angle $\pi/2$; the tetrahedron has edge 2.039648903 and an irrational multiple — the Dehn invariant vanishes for one and not for the other (Section 5). Fifth, in the plane the moves can actually be exhibited. Two triangles sharing a base and differing only in where the apex sits were each cut into three pieces and laid out as the same rectangle — the area stayed at 1.0 and all 4000 random points fell inside exactly one piece (Section 6).

Scope of this paper (scope note): **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED** — Hilbert's third problem, the Dehn invariant, the Wallace–Bolyai–Gerwien theorem and Sydler's theorem are all standard. Dehn's theorem is not proved — what is checked here is that the dihedral angle is not a rational multiple of a half turn, and that the invariant survives cutting is cited, not shown. The Dehn invariant is not written down as a single number — it lives in $\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q})$, so what can be written is only that it is not zero. Nothing is claimed about five dimensions or more. Curved solids and infinitely many pieces are not treated — the pieces are finite in number and the cuts are flat. Relation to Paper 179: 179 treated the distinct roots of “cannot be constructed” — the impossibility here has nothing to do with the axiom of choice, and is on the side where the obstruction can be computed by a finite procedure. What is added is setting the five dihedral angles side by side and finding exactly one rational multiple, settling the irrationality with an integer recurrence and residues modulo three, measuring how close it comes out to 200000, writing out the two solids of equal volume whose invariants differ, and constructing the plane dissection and counting the overlaps.

1. Notation and Position

Two polyhedra are **scissors-congruent** when one can be cut into finitely many polyhedra which can be moved to build the other. If they are, their volumes are equal. The question is the converse: **if the volumes are equal, can it always be done?**

In the plane the answer is **yes** (Wallace–Bolyai–Gerwien). In space it is **no** (Dehn). This paper measures the difference.

Position: In Paper 203 the counts of critical points moved a great deal while the alternating sum never moved — here the number of pieces can be anything at all while the volume and one other thing never move. In Paper 208 dimension and curvature did the same work — here the dimension itself is the fence.

2. Only the Cube Has a Rational Multiple

Check 2.1: divide the dihedral angle of each regular solid by a half turn.

Solid	Dihedral angle $\div \pi$	Form
Cube	0.5 (exactly)	$\pi/2$
Tetrahedron	0.391826552030607	$\arccos(1/3)$
Octahedron	0.608173447969393	$\arccos(-1/3)$
Dodecahedron	0.647583617650433	$\arccos(-1/\sqrt{5})$
Icosahedron	0.76772047280123	$\arccos(-\sqrt{5}/3)$

Observation 2.2

Of the five, exactly one gives a rational number. And that one is also the only regular solid that fills space without gaps — though this paper does not show that the coincidence is no coincidence.

3. Integers Alone Settle It

Put $\theta = \arccos(1/3)$. Since $\cos((n + 1)\theta) = \frac{2}{3}\cos(n\theta) - \cos((n - 1)\theta)$, writing $\cos(n\theta) = a_n/3^n$ gives the integer recurrence

$$a_{n+1} = 2a_n - 9a_{n-1}$$

with the initial values $a_0 = a_1 = 1$.

Check 3.1:

n	a_n	$a_n \bmod 3$	$2^{n-1} \bmod 3$
1	1	1	1
2	−7	2	2
3	−23	1	1
4	17	2	2
5	241	1	1
10	57113	2	2

From $n = 1$ to 199, a_n was never divisible by three, and $a_n \bmod 3$ never differed from $2^{n-1} \bmod 3$. The values agreed with multiprecision $\cos(n\theta)$ to within 10^{-41} .

Observation 3.2

Modulo three the recurrence reads $a_{n+1} \equiv 2a_n$, and $a_1 = 1$, so $a_n \equiv 2^{n-1}$ — and 2^{n-1} is never divisible by three, so neither is a_n . For $\cos(n\theta) = \pm 1$ with $n \geq 1$ one would need $a_n = \pm 3^n$, which needs three to divide a_n . So $n\theta$ is never a whole number of half turns, and θ/π is irrational.

Observation 3.3

What does the work here is one single property of an integer: that it is not divisible by three. A question about geometry turned into a question about a remainder — and no approximation could ever have settled it: however far the precision is pushed, all one can say is that it has not arrived yet.

4. It Comes Close and Never Arrives

Check 4.1: measure how far $n\theta/\pi$ lies from the nearest whole number, for n up to 200000.

n	Distance to the nearest whole number
1	0.391826552
2	0.2163468959
3	0.1754796561
5	0.04086723985
38882	$3.945928121 \times 10^{-6}$

Observation 4.2

It comes within four parts in a million. If “close” were enough, then 38882 of these angles laid end to end would almost be a whole number of half turns. But cutting and reassembling demands exactly, not almost — and what the previous section says is that this sequence never touches zero.

5. The Core: Equal Volume, Different Invariant

The **Dehn invariant** of a polyhedron P collects the edge lengths ℓ_i and dihedral angles θ_i as

$$D(P) = \sum_i \ell_i \otimes \theta_i \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q}),$$

and it does not change under cutting and reassembling.

Check 5.1: scale both to volume exactly 1.

Solid	Edge	Edges	Total edge length	Dihedral $\div \pi$
Cube	1.0	12	12.0	0.5 (rational)
Tetrahedron	2.039648903	6	12.23789342	0.391826552031 (irrational)

Observation 5.2

For the cube every dihedral angle is a rational multiple of π , hence zero in $\mathbb{R}/\pi\mathbb{Q}$, so every term vanishes and the sum is zero. For the tetrahedron the angle is not a rational multiple and the edge length is not zero, so the sum is not zero. The volumes are equal and the invariants differ — so no dissection into finitely many pieces can carry either into the other.

Observation 5.3

One thing should be said plainly here. Because $D(P)$ lives in $\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q})$, it cannot be written down as a single real number — turning the tensor product into numbers requires choosing a basis over $\mathbb{Q}$, and that requires the axiom of choice. What this paper could compute is only that it is not zero, and for that a remainder of an integer sufficed. This is a distinct root from Paper 179: that one is about the side where choice is needed, and the obstruction here can be checked by a finite procedure.

6. In the Plane the Moves Can Be Exhibited

Fix the base at $(0,0)$ – $(2,0)$ and take two triangles differing only in the x -coordinate of the apex. Both have area 1. Cutting along the midline, and then along the perpendicular dropped from the apex, gives **three** pieces; rotating two of them a half turn about the midpoints builds the same rectangle, **2** by $\frac{1}{2}$.

Check 6.1:

Apex x	Triangle area	Sum of pieces	Sum after moving	Points in exactly one
0.3	1.0	1.0	1.0	4000/4000
1.4	1.0	1.0	1.0	4000/4000

(Four thousand points were taken at random inside the rectangle and counted against the three pieces.)

Observation 6.2

There is neither overlap nor gap. And since both triangles arrive at the same rectangle, each can be carried into the other. The same rectangle comes out wherever the apex is put, so all triangles on a given base and height are one single family.

Observation 6.3

In the plane one can show the moves with one's hands. In space one can only show with a number that there are none — and “none have been found” is not “there are none”: Section 5 says the second.

7. How Many Rails the Fence Has

Census 7.1:

Dimension	What suffices	Name
2	equal area (one rail)	Wallace–Bolyai–Gerwien
3	volume and invariant (two rails)	Dehn, Sydler
4	two are known to suffice as well	Jessen
5 and up	open	—

Observation 7.2

Equal area sufficed in the plane because there was nothing besides area that survived the cutting. In space there is one more thing — and exactly one more (Sydler). Paper 198 also had two fences — the shape is the same and the contents are different.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates “equal volume” from “can be cut and reassembled” is the dimension. At $d = 2$ they are the same phrase, and at $d = 3$ they are two different phrases. It is not a matter of precision, nor of skill with a knife — it is not that someone who can cut in the plane cannot cut in space, but that in space there is one more quantity that cutting does not destroy.

Observation 8.2 (the fence)

The fence around “equal volume means they can be carried into one another” is that $d \leq 2$. Outside the fence, the condition stays necessary and stops being sufficient — it does not become false: it becomes not enough.

Observation 8.3

In Paper 203 the counts moved and the alternating sum did not — here the number of pieces moves and the volume and the invariant do not. Both look for what disappears under moving in order to find what does not. And where 203 needed one unmoving quantity, this needs two — how many are needed is not known until they are counted.

9. The Census and the Close

What this paper measured:

1. Exactly one rational multiple (Check 2.1): of five regular solids, only the cube.

2. **Integers settle the irrationality** (Check 3.1): $a_{n+1} = 2a_n - 9a_{n-1}$, never divisible by three out to 199.
3. **Close but never arriving** (Check 4.1): 3.9×10^{-6} at $n = 38882$.
4. **Equal volume, different invariant** (Check 5.1): zero for the cube, not zero for the tetrahedron.
5. **In the plane the moves are visible** (Check 6.1): three pieces, 4000/4000 in exactly one.
6. **One rail in two dimensions, two in three** (Census 7.1).
7. **The separator is the dimension** (Observation 8.1).

Close: On paper, matching areas is always enough to cut and rearrange — and one can show the moves by hand. The moment the same sentence is tried in space, one further quantity appears that cutting does not destroy. And the evidence that it does not vanish was not to be found anywhere in the geometry — it is one single property of an integer, that a_n is not divisible by three. Since it comes within four parts in a million, no measurement could have told. What separated “not yet found” from “does not exist” was not precision but a remainder.

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