

Adding More Terms Takes You Further from the Answer

— Converging and Approximating Are Two Different Limits, and an Asymptotic Series Is Best at Its $1/x$ -th Term: the Error There Is Smaller Than Every Power of x — [Paper 211]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

Adding the terms of a series brings you nearer the answer — and that is true only for one kind of series. This paper measures a series that diverges at every x and still gives twenty-one digits, and writes in numbers that converging and approximating are claims about different limits. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, a convergent series behaves as one expects. The error of the series for e falls from 0.218282 to 2.05×10^{-20} without once turning back (Section 2). Second, the expansion of $E(x) = \int_0^\infty e^{-t}/(1+xt) dt$ diverges at every x . At $x = 0.1$ the error falls to 1.770×10^{-4} at ten terms and then climbs to 65.9 at thirty and 1.63×10^7 at forty (Section 3). Third, there is a rule for where to stop. The best number of terms at $x = 0.2, 0.1, 0.05, 0.02, 0.01$ is 5, 10, 20, 50, 100, which is exactly $1/x$, and the error there is close to $e^{-1/x}$ (Section 4). Fourth, Stirling's series shows the same shape. The best number of terms at $z = 2, 5, 10, 20$ is 7, 16, 32, 63, which matches πz — and the error at $z = 20$ is 9.43×10^{-57} (Section 6). Fifth, this is the core. Holding x and adding terms diverges; holding the number of terms and letting x fall sends the error over x^2 to zero and the error over x^3 to 6.0 — and $6 = 3!$ is the next coefficient itself (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Poincaré’s definition of an asymptotic expansion, the divergence of Stirling’s series and optimal truncation are all standard. Neither Borel summation nor hyperasymptotics is treated — nothing is said about getting past the $e^{-1/x}$ wall. The Stokes phenomenon is not treated. The divergence of perturbation theory is not discussed — every term of the series here is finite. Relation to Papers 35 and 39: where those said that divergences worsen at higher order, they meant that the individual terms diverge, and this paper is about a series that diverges although every term is finite — “diverges” has distinct roots too. Relation to Paper 197: 197 treated “converges” as four separate claims — what is separated here is not inside those four but convergence and asymptoticity, which are different limits to begin with. What is added is measuring at five values of x that the best number of terms is $1/x$, seeing that the error there is close to $e^{-1/x}$, showing by ratios that this error is smaller than every power of x , confirming at four values of z that the best number of terms in Stirling’s series matches πz , and running the two limits separately on one series, watching one diverge while the other settles on $3!$.

1. Notation and Position

A series $\sum_n a_n x^n$ converges to f when, holding x and letting the number of terms N grow, the partial sums approach $f(x)$. It is an asymptotic expansion when, holding N and letting x tend to zero,

$$f(x) - \sum_{n=0}^{N-1} a_n x^n = O(x^N)$$

(Poincaré 1886). The variable being moved is not the same.

Position: In Paper 197 “converges” was four separate claims — here, outside those four, converging and approximating turn out to be different limits. In Paper 168 what Lebesgue bought was closure under limits — here too the question is which limit.

2. A Convergent Series Behaves as One Expects

Check 2.1: partial sums of $e = \sum 1/n!$.

Terms	Partial sum	Error
3	2.5	0.218282
5	2.7083333333333333	0.0099485
9	2.7182787698412698413	3.05862×10^{-6}
13	2.7182818282861685639	1.72877×10^{-10}
21	2.7182818284590452353	2.0503×10^{-20}

Every term helps and the error never turns back. This is the picture that ordinarily comes with the phrase “adding up a series”.

3. A Series That Diverges at Every x

Consider $E(x) = \int_0^\infty \frac{e^{-t}}{1+xt} dt$. Expanding $1/(1+xt)$ and integrating term by term gives $\sum_n (-1)^n n! x^n$, whose terms $n! x^n$ grow without bound for every $x \neq 0$.

Check 3.1: at $x = 0.1$, the true value being **0.9156333393978808187606982** by quadrature.

Terms	Size of the term	Partial sum	Error
3	0.02	0.92	0.00436666
8	0.000504	0.915416	0.000217339
10	0.00036288	0.91545632	0.000177019
12	0.000399168	0.915420032	0.000213307
20	0.0121645100409	0.9076137084	0.00801963
30	88.4176199374	-64.9806812262	65.8963
40	20397882.0812	-16252520.1096	1.62525×10^7

Observation 3.2

The error falls, meets a floor, and then climbs without limit. After forty terms the answer agrees with the truth in not one digit — where stopping at three would have given four.

4. There Is a Rule for Where to Stop

Check 4.1:

x	Best number of terms	Error there	$e^{-1/x}$
0.2	5	0.018289119	0.006737947
0.1	10	0.0001770194	4.539993×10^{-5}
0.05	20	1.14578×10^{-8}	2.0611536×10^{-9}
0.02	50	$1.7036434 \times 10^{-21}$	$1.9287498 \times 10^{-22}$
0.01	100	$4.6546742 \times 10^{-43}$	3.720076×10^{-44}

Observation 4.2

The best number of terms is 5, 10, 20, 50, 100, which is exactly $1/x$. And the error there stays within a small factor of $e^{-1/x}$. The place where the term $n!x^n$ is smallest is exactly the place to stop.

Observation 4.3

There is no room here for wanting more accuracy. The moment x is fixed, the accuracy that can be reached is fixed with it — and no amount of further adding gets past it.

5. And That Error Is Smaller Than Every Power of x

Check 5.1:

x	$e^{-1/x}$	x^5	Ratio
0.2	0.006737946999	0.00032	21.05608437
0.1	$4.539992976 \times 10^{-5}$	1.0×10^{-5}	4.539992976
0.05	$2.061153622 \times 10^{-9}$	3.125×10^{-7}	0.006595691592
0.02	$1.928749848 \times 10^{-22}$	3.2×10^{-9}	$6.027343275 \times 10^{-14}$

Observation 5.2

The ratio goes to zero. Replacing the five by any other number does the same — because $e^{-1/x}$ shrinks faster than every power of x . So this error appears at no order of the series at all. It is a quantity that computing to higher order cannot in principle reveal.

6. Stirling's Series Shows the Same Shape

The expansion $\log \Gamma(z) \sim (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{n \geq 1} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}$ diverges as well.

Check 6.1:

z	Best number of terms	πz	Error there
2	7	6.28	3.9003289×10^{-7}
5	16	15.71	1.589869×10^{-15}
10	32	31.42	$2.5831777 \times 10^{-29}$
20	63	62.83	$9.4340912 \times 10^{-57}$

Observation 6.2

The best number of terms matches πz , and the error there is of the size of $e^{-2\pi z}$. This series has nothing to do with the one in Section 4, and the same shape came out — both obey the single rule of stopping where the terms are smallest.

7. The Core: One Word for Two Different Limits

Run the two limits separately on one and the same series.

Check 7.1 (hold x , add terms): at $x = 0.1$ the errors at 10, 20, 30, 40 terms are

$$1.77 \times 10^{-4}, \quad 8.02 \times 10^{-3}, \quad 65.9, \quad 1.63 \times 10^7.$$

It does not converge.

Check 7.2 (hold the terms, lower x): keeping three terms, $n = 0, 1, 2$.

x	Error	Error / x^2	Error / x^3
0.2	0.02788911858	0.69722796	3.486139822
0.05	0.0006290900808	0.25163603	5.032720646
0.01	$5.771326698 \times 10^{-6}$	0.057713267	5.771326698
0.005	$7.353641287 \times 10^{-7}$	0.029414565	5.882913029

Observation 7.3

The middle column goes to zero and the right one settles on six. And $6 = 3!$ is the coefficient of the very term that was not added. So the series is perfectly correct as an asymptotic expansion — while diverging.

Observation 7.4

Converging and approximating are claims about different limits. One holds x and moves N ; the other holds N and moves x . A series may be both, one, or neither — and it is the second that decides whether it is any use.

8. The Separator and the Fence

Check 8.1: fifty terms each, and how far they get.

Series	Error at fifty terms
The expansion of $E(0.02)$ (divergent everywhere)	1.70×10^{-21}
The alternating harmonic series for $\log 2$ (convergent)	9.90×10^{-3}

(Different functions, so this is **an illustration and not a comparison.**)

Observation 8.2 (the separator)

What decides whether a series is usable is not whether it converges — it is which variable one intends to move. If x is settled and small, the divergent series is faster. If x is to be moved, convergence is needed. To separate them, say which limit is intended.

Observation 8.3 (the fence)

The fence around “more terms make it better” is that the series converge. Outside the fence the sentence becomes false — it gets better and then worse. And it worsens sharply: forty terms at $x = 0.1$ miss by 1.6×10^7 .

Observation 8.4

In Paper 209 it was a remainder of an integer that guaranteed the approach never arrives — here the opposite happens: it arrives, and then departs again. In Paper 210 four quantities shared one name — here two limits share the one word “approaches”.

9. The Census and the Close

What this paper measured:

1. A convergent series is well behaved (Check 2.1): the error never turns back.
2. A divergent one climbs again (Check 3.1): 1.63×10^7 at forty terms.
3. The best number of terms is $1/x$ (Check 4.1): 5, 10, 20, 50, 100.
4. The error is smaller than every power of x (Check 5.1): the ratio falls to 6.03×10^{-14} .

5. For Stirling it is πz (Check 6.1): **7, 16, 32, 63**.

6. The two limits are different (Checks 7.1 and 7.2): **divergent, and yet the ratio settles on 3!**.

7. Convergence buys nothing (Check 8.1): **fifty terms give 10^{-21} against 10^{-2}** .

Close: An answer that had four correct digits at three terms has none at all by forty. And still the series is not wrong — as an asymptotic expansion it gets even the 3! of the next coefficient exactly right. What was wrong was not the series but the assumption that adding makes it better. There is a rule for where to stop, the accuracy reached there is $e^{-1/x}$, and that is smaller than every power of x — which is to say it is written at no order of the series at all. What cannot be seen is not thereby absent: it merely has no place to sit in that way of speaking.

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