

Every Map Must Distort Something

— No Projection Preserves Both Angle and Area: Whatever Projection Is Drawn, the Gaussian Curvature Measured from the Map's Own Metric Is Still 1 — [Paper 212]

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— *From thinking to syncing.* —

ABSTRACT

A map carrying the sphere to the plane must give up angle or area, or both. This paper writes that “must” in numbers and shows that what forces the sacrifice is a single number. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, Mercator keeps every angle exactly and gives up area. The area scale is **2.0** at latitude 45° , **4.0** at 60° and **131.6** at 85° (Section 2). Second, Lambert keeps every area exactly and gives up angle. The area scale is one at every latitude, to within 1.15×10^{-41} , while the angular deformation reaches **73.74°** at 60° and **160.08°** at 85° (Section 3). Third, this is the core. Writing the metric of the sphere in the coordinates of each of the three maps and computing the Gaussian curvature from that metric alone gave exactly one, in all three, at every point — three metrics that look nothing alike return the same single number (Section 4). Fourth, a cylinder returns zero. That is why paper can be rolled into a cylinder or a cone without any stretching at all, and never into a sphere (Section 5). Fifth, small maps do not suffer. The area error of a map one kilometre across is $5.08 \times 10^{-7} \%$, and **0.00508 %** at 111 kilometres, but **57.08 %** for a hemisphere (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Gauss's Theorema Egregium, Tissot's indicatrix, and the incompatibility of conformality with equal area are all standard. The Theorema Egregium is not proved — this paper measures its consequences and cites the theorem itself. No survey of real map projections is offered — only three are treated. The Earth is taken to be a sphere — neither its flattening nor its terrain enters. No projection is called good — what is written is only what each keeps and what each gives up. Relation to Paper 91: 91 was a paper about the vocabulary of curvature being intrinsic — this one counts the consequence on a concrete object, a map. Relation to Paper 198: 198 said that a flat cylinder has no conjugate point at all — this paper says that the same flatness is why paper rolls. Two consequences of one fact. What is added is measuring the scale factors and angular deformations of three projections latitude by latitude, computing the Gaussian curvature from each map's own metric and obtaining the same one, obtaining zero for the cylinder, tabulating the area error against the size of the map at five scales, and computing the shortening of the rim needed to make a cone.

1. Notation and Position

Consider a map from the unit sphere to the plane. Its differential at each point is a linear map; let σ_1, σ_2 be its two singular values, the scale factors along the parallel and along the meridian. Then

$$\text{area scale} = \sigma_1 \sigma_2, \quad \sin \frac{\omega}{2} = \frac{|\sigma_1 - \sigma_2|}{\sigma_1 + \sigma_2},$$

where ω is the greatest angular deformation. **Conformal** means $\sigma_1 = \sigma_2$; **equal-area** means $\sigma_1 \sigma_2 = 1$.

Position: In Paper 203 the counts of critical points moved and the alternating sum did not — in Paper 209 the dissection could be changed at will and the Dehn invariant did not move. Here the map may be drawn any way at all and the curvature does not move. All three have one shape: what does not move is what settles what cannot be done.

2. Mercator Keeps the Angles and Gives Up Area

Check 2.1: Mercator, $x = \lambda$ and $y = \log \tan(\pi/4 + \varphi/2)$.

Latitude	σ_1	σ_2	Area scale	Angular deformation
0°	1.0	1.0	1.0	0.0°
45°	1.4142136	1.4142136	2.0	0.0°
60°	2.0	2.0	4.0	0.0°
75°	3.8637033	3.8637033	14.928203	0.0°
85°	11.473713	11.473713	131.6461	0.0°

Observation 2.2

The angular deformation is exactly zero at every latitude. In exchange the area swells to a hundred and thirty times — a country at 85° is drawn a hundred and thirty times too large in area against one on the equator. This is not an error but a payment: every angle was kept, and all of the area was handed over.

3. Lambert Keeps the Areas and Gives Up Angle

Check 3.1: the Lambert cylindrical equal-area projection, $x = \lambda$ and $y = \sin \varphi$.

Latitude	σ_1	σ_2	Area scale	Angular deformation
0°	1.0	1.0	1.0	0.0°
30°	1.1547005	0.8660254	1.0	16.426421°
45°	1.4142136	0.70710678	1.0	38.942441°
60°	2.0	0.5	1.0	73.739795°
85°	11.473713	0.087155743	1.0	160.07572°

(The largest departure of the area scale from one was 1.15×10^{-41} , which is the floor of the arithmetic.)

Check 3.2: the worst case out to latitude 85°, for all three maps.

Map	Largest area error	Largest angular deformation
Mercator	130.6461	0.0°
Lambert	1.15×10^{-41}	160.07572°
Equirectangular	10.473713	114.20906°

Observation 3.3

One column or the other is always large. A map making both zero would have $\sigma_1 = \sigma_2 = 1$, and that is an isometry — it would mean laying the sphere on the plane without stretching. The equirectangular projection keeps neither: splitting the difference only makes both worse.

4. The Core: Whatever Map Is Drawn, the Curvature Is Still One

Write the metric of the sphere in the coordinates of each map.

Map	The sphere's metric in those coordinates
Equirectangular	$ds^2 = \cos^2 y \, dx^2 + dy^2$
Mercator	$ds^2 = \operatorname{sech}^2 y \, (dx^2 + dy^2)$
Lambert	$ds^2 = (1 - y^2) \, dx^2 + dy^2 / (1 - y^2)$

For an orthogonal metric $ds^2 = E \, du^2 + G \, dv^2$,

$$K = -\frac{1}{2\sqrt{EG}} \left[\frac{\partial}{\partial v} \left(\frac{\partial E / \partial v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left(\frac{\partial G / \partial u}{\sqrt{EG}} \right) \right],$$

which is built from the metric alone.

Check 4.1:

Map	Point	Curvature from the metric	Departure from one
Equiarectangular	(0.3, 0.2)	1.0	0.0
Equiarectangular	(1.1, −0.55)	1.0	0.0
Mercator	(0.3, 0.2)	1.0	0.0
Mercator	(1.1, −0.55)	1.0	0.0
Lambert	(0.3, 0.2)	1.0	0.0
Lambert	(1.1, −0.55)	1.0	0.0

Observation 4.2

The three metrics look nothing alike. One carries a $\cos^2$, one a sech^2 , one a $1 - y^2$ and its reciprocal. And the number that the same procedure returns from them is one, at every point — change the drawing however you like, and this number cannot be moved.

Observation 4.3

Here is where the “must” comes from. A map both conformal and equal-area would be an isometry, carrying the plane's curvature of zero onto the sphere — but the curvature is settled by the metric alone, and it is one. And $0 \neq 1$. This is not a matter of skill: no amount of ingenuity gets past it.

5. A Cylinder Returns Zero, Which Is Why Paper Rolls

Check 5.1: a cylinder in its own coordinates has $ds^2 = dx^2 + dy^2$, and the same procedure returns **0.0** at (0.3, 0.2) and at (1.1, −0.55).

Observation 5.2

That is the same number as a flat sheet. So paper rolls into a cylinder or a cone with no stretching whatever — and never into a sphere. A cylinder is “curved seen from outside” and “flat measured from inside”: “curved” has distinct roots too.

Observation 5.3

From this it follows that curving a sheet one way leaves it no choice but to stay straight the other way — curving both ways would make $K \neq 0$, which cannot be had without stretching. Paper 198 saw that a flat cylinder has no conjugate point at all — another consequence of the same flatness.

6. Small Maps Do Not Suffer

Draw, for a cap of angular radius θ , the map that keeps distance from one point exactly. Its area scale at the rim is $\theta / \sin \theta$.

Check 6.1:

Angular radius	On the globe	Area scale at the rim	Excess
0.01°	about 1 km	1.00000000508	$5.08 \times 10^{-7} \%$
1°	about 111 km	1.00005077137	0.00508 %
10°	about 1100 km	1.00509505798	0.5095 %
30°	about 3300 km	1.0471975512	4.7198 %
90°	a hemisphere	1.57079632679	57.0796 %

Observation 6.2

A town plan is wrong by five parts in a thousand million. So daily life cannot notice that the Earth is a sphere — and not noticing is not the same as there being nothing there. Widen it to a hemisphere and it is 57 %: the error grows with the area and never quite vanishes.

7. What a Tailor Already Knows

Lay a flat disc of radius a onto a sphere of radius R . Its rim must shrink from $2\pi a$ to $2\pi R \sin(a/R)$.

Check 7.1:

a	R	a/R	Fraction of rim to be gathered
1	20	0.05	0.0417 %
5	20	0.25	1.0384 %
10	20	0.5	4.1149 %
10	10	1.0	15.8529 %

Observation 7.2

That shortfall is curvature made visible. Darts, the gores of a globe and the peel of an orange are all payments of this one number. And there is no way of not paying — gather, cut, or stretch: those are the three.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of “curved” is whether one looks from outside or measures from inside. A cylinder is curved from outside and flat from inside — a sphere is curved either way. Only the inside one bears on maps.

Observation 8.2 (the fence)

The fence around “can be laid out on the plane” is that the Gaussian curvature be zero. Inside the fence, cylinder, cone and plane, it can be done perfectly; outside it, no skill suffices. This is not a question of accuracy: it is the difference between nought and one.

Observation 8.3

The shape is that of Papers 203 and 209 — in both, a quantity that no effort can move settled the boundary of what could be done. With one difference: there the invariant told two objects apart, and here it runs through every way of drawing one object. “Invariant” too is unsettled until one says invariant under what.

9. The Census and the Close

What this paper measured:

1. Mercator keeps angle and gives up area (Check 2.1): **131.6 times at 85° .**
2. Lambert keeps area and gives up angle (Check 3.1): **73.74° at 60° .**
3. Splitting the difference worsens both (Check 3.2).
4. The curvature is one on every map (Check 4.1): **three metrics, six points, all.**
5. A cylinder gives zero (Check 5.1): **which is why paper rolls.**
6. Small maps do not suffer (Check 6.1): **$5 \times 10^{-7} \%$ at a kilometre, 57% for a hemisphere.**
7. A cone needs **4.11% of its rim gathered** (Check 7.1).

Close: A world map lies not because the maker is unskilled. Whatever projection is drawn, the Gaussian curvature measured from that map's own metric is one — and a flat sheet's is nought. What is being paid is the difference between those two numbers, and only the manner of payment can be chosen: in angle, in area, or a little of both. There is no path that does not pay. And on the scale of a town the bill falls to five parts in a thousand million, so daily life never receives it — and a map turns into a lie only when the absence of a bill is read as the absence of a debt.

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