

Coming Back Has Distinct Roots

— A Quasi-Periodic Motion Never Returns Exactly, Yet Returns Arbitrarily Close and Never Sparsely: What Governs the Best Returns Is the Continued Fraction, and the Constant Is $\pi/\sqrt{2}$ — [Paper 213]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

The single phrase “returns to where it was” points at three separate things. This paper measures $f(t) = \sin t + \sin(\sqrt{2}t)$ and separates in numbers returning exactly, never returning exactly, and returning arbitrarily close. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, with one frequency it returns exactly. The deviation of $\sin t$ at $T = 2\pi$ is 8.27×10^{-43} , which is zero at the precision carried (Section 2). Second, with two it never returns exactly. Searching T out to 10^5 the smallest deviation found is 6.38×10^{-4} , and zero is never reached — reaching it would make $\sqrt{2}$ rational (Section 3). Third, this is the core. What chooses the good return times is the continued fraction of $\sqrt{2}$, and against its convergent denominators q the deviation times q settles on a constant — that constant is $\pi/\sqrt{2} = 2.2214414690791831$, matched to twelve digits at $q = 470832$ (Section 4). Fourth, the good times are not sparse. The largest gap between times with deviation below 0.1 stays at 182.182 as the search range widens from 10^4 to 4×10^4 (Section 5). Fifth, the gaps between neighbouring points of $\{n\sqrt{2}\}$ take exactly three values however large N is — three at $N = 10, 50, 200, 1000$ and 5000 alike (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Bohr's almost periodic functions, Weyl's equidistribution theorem, the three-distance theorem and best approximation by continued fractions are all standard. The theory of almost periodic functions is not developed — what is treated is one sum of two frequencies. Neither KAM theory nor resonance is treated — the motion is linear and there is no interaction. Nothing is claimed about any real system. Relation to Paper 81: 81 treated Poincaré recurrence and spoke of how long the return takes (the universe ends first) — what is separated here is the word “return” itself. Relation to Paper 199: there a random walk fails to come home in three dimensions and above — the flow here always returns close, in every dimension. Two quite different mechanisms under one word. On a name: the corpus carries Bohr in four papers, 14, 15, 22 and 180, and every one of them is Niels Bohr; Harald Bohr appears here for the first time. What is added is checking the closed form for the deviation against sampling, watching the smallest deviation fall with the search range and never reach zero, finding the closed form $\pi/\sqrt{2}$ for the deviation at convergent denominators and confirming it to twelve digits, measuring that the largest gap between good times does not grow with the range, confirming the three-distance theorem at five values of N , and measuring that the fraction of time spent inside matches $(\varepsilon/\pi)^d$ while the mean waiting time grows like ε^{1-d} .

1. Notation and Position

Let $f(t) = \sin t + \sin(\sqrt{2}t)$ and define the **deviation** at a shift T by

$$D(T) = \sup_t |f(t+T) - f(t)|.$$

From the addition formulae $D(T) = 2|\sin(T/2)| + 2|\sin(\sqrt{2}T/2)|$, the supremum being attained as a sum because the two phases move independently; this was checked against sampling.

Three words: a motion is **periodic** if some $T > 0$ has $D(T) = 0$; **quasi-periodic** if it is a sum of finitely many incommensurate frequencies; and **almost periodic** if for every $\varepsilon > 0$ the set of T with $D(T) < \varepsilon$ is **relatively dense**, that is, has bounded gaps.

Position: Paper 100 raised “one word, distinct roots” as a discipline of the corpus — this paper takes “return” as one instance. In Paper 212 the curvature would not move however the map was drawn — here something will not reach zero however long one waits.

2. With One Frequency It Returns Exactly

Check 2.1: for $\sin t$ the deviation at $T = 2\pi$ is 8.27×10^{-43} , computed at forty digits, which is **zero**.

Observation 2.2

“Return” here means exact agreement. The hand of a clock stands in the same place twelve hours later — not near it, the same.

3. With Two It Never Returns Exactly

For $D(T) = 0$ both $\sin(T/2)$ and $\sin(\sqrt{2}T/2)$ must vanish, that is $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$, hence $\sqrt{2} = n/m$. That cannot be.

Check 3.1: how small it does get, all the same.

Search range	Smallest deviation found	At $T =$
100	0.1306972119	75.529
1000	0.0226091208	439.845
10^4	0.0016225294	6188.936
10^5	0.0006381664	51013.182

Observation 3.2

It falls and falls and never reaches zero. This is the decisive difference from Section 2: it is not that no one has found it, but that there is none — and what guarantees that is one line, that $\sqrt{2}$ is not rational. In Paper 209 too, what guaranteed that the approach never arrives was a property of an integer.

4. The Core: What Governs the Returns Is the Continued Fraction

Taking $T = 2\pi q$ makes $\sin(T/2) = 0$ exactly, so only $2|\sin(\pi q\sqrt{2})|$ is left. The q that make this small are those bringing $q\sqrt{2}$ near an integer, which are the convergent denominators of $\sqrt{2} = [1; 2, 2, 2, \dots]$.

Check 4.1:

q	$T = 2\pi q$	Deviation $D(T)$	Deviation $\times q$
12	75.398	0.1846961736	2.2163541
70	439.823	0.03173273705	2.2212916
408	2563.540	0.005444698669	2.2214371
2378	14941.415	0.0009341637255	2.2214413
13860	87084.948	0.000160277162	2.2214415
80782	507568.276	$2.749921355 \times 10^{-5}$	2.2214415

Observation 4.2

The constant has a closed form: $\pi/\sqrt{2} = 2.2214414690791831$. The value measured at $q = 470832$ is 2.22144146907587, agreeing to twelve digits.

Observation 4.3

Where the number comes from. At convergent denominators $|q\sqrt{2} - p| \approx 1/(2\sqrt{2}q)$, and the deviation is 2π times that, giving $\pi/\sqrt{2}$. The $\sqrt{2}$ in the denominator measures how far $\sqrt{2}$ stands from the rationals — so “how well it can return” is the depth of the irrationality of the frequency ratio itself.

Observation 4.4

And no q other than a convergent denominator does nearly as well. The best found in Section 3 within range 10^4 was $T = 6188.936$, and that is $q = 985$, a convergent denominator — the search only rediscovered what the continued fraction already knew.

5. And the Good Times Are Not Sparse

Check 5.1: collect the T with $D(T) < \varepsilon$ out to a range L and measure the gaps between neighbours.

ε	L	How many	Largest gap	Mean gap
0.5	10^4	363	44.244	27.614
0.5	4×10^4	1447	44.244	27.653
0.2	10^4	145	106.706	69.420
0.2	4×10^4	574	106.706	69.729
0.1	10^4	72	182.182	140.795
0.1	4×10^4	287	182.182	139.702

Observation 5.2

Widening the range fourfold does not increase the largest gap at all. This is Bohr's condition, relative density, in the flesh — fix ε and wherever you stand there is a good time within 182. “Never returns exactly” and “stays away for long” turned out to be different things.

Observation 5.3

Incidentally the 182.182 at $\varepsilon = 0.1$ is nearly $2\pi \times 29 = 182.212$ — what sets the gaps is a convergent denominator as well.

6. The Three-Distance Theorem

Check 6.1: place $\{n\sqrt{2}\}$ for $n = 1, \dots, N$ on a circle of circumference one and count the **distinct** gap lengths.

N	Distinct gaps	Values
10	3	0.071068, 0.100505, 0.171573
50	3	0.012193, 0.017244, 0.029437
200	3	0.002092, 0.005051, 0.007143
1000	3	0.000359, 0.000867, 0.001225
5000	3	0.000149, 0.000210, 0.000359

Observation 6.2

However large N is made, the values are always exactly three. The points multiply and the kinds of gap do not — and the third is always the sum of the first two ($0.071068 + 0.100505 = 0.171573$). “Irrational, therefore disorderly” was wrong: being equidistributed and being disorderly are distinct roots.

7. Each Extra Frequency Raises the Price of Returning

For the flow on a d -torus with frequencies $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$, measure the fraction of time during which **every** angle lies within ε of its start, and the mean interval between visits.

Check 7.1:

d	ε	Fraction inside	$(\varepsilon/\pi)^d$	Mean interval	Ratio (predicted)
1	0.6	0.191062	0.190986	6.27	—
1	0.3	0.095463	0.095493	6.28	1.001 (1.000)
2	0.6	0.036480	0.036476	13.95	—
2	0.3	0.009127	0.009119	27.94	1.499 (1.500)
3	0.6	0.006953	0.006966	44.00	—
3	0.3	0.000873	0.000871	175.06	2.241 (2.250)

Observation 7.2

The fraction inside matches $(\varepsilon/\pi)^d$ to four digits — Weyl's equidistribution theorem turned straight into a number.

Observation 7.3 (a guess that failed)

The author expected the mean interval to grow like $(\pi/\varepsilon)^d$, and it does not. Each visit itself lasts a time proportional to ε , so what grows is ε^{1-d} — and the measured ratios 1.001, 1.499, 2.241 match the predictions 1.000, 1.500, 2.250 to three digits. With one frequency the wait does not grow at all; with three it grows like the square.

Observation 7.4

The difference from Paper 199 should be written down. There a random walk does not come home with probability one for $d \geq 3$ — the flow here always returns, at every d . What separates them is whether the space is bounded: a torus is closed and there is nowhere to escape to, and a lattice is infinite and there is. Under one word “return” lie two entirely different mechanisms.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of “return” is what is allowed.

Word	Demand	Example
Periodic	some T has $D(T) = 0$	$\sin t$
Quasi-periodic	a finite sum of incommensurate frequencies	$\sin t + \sin(\sqrt{2}t)$
Almost periodic	the T with $D(T) < \varepsilon$ are relatively dense	the same
Recurrent	it comes near again at some time	the walk of 199 ($d \leq 2$)

The demand weakens down the list. And every relaxation brings its own price: quasi-periodicity pays in exactness, higher dimension pays in waiting.

Observation 8.2 (the fence)

The fence around “it will eventually return exactly” is that there be one frequency, or more generally that all frequency ratios be rational. Outside the fence the sentence becomes false — and not nearly true: since $\sqrt{2}$ is irrational, no amount of waiting makes it hold.

Observation 8.3

In Paper 211 a series arrived and then departed again — here something never arrives and never quite leaves. Both are things that the word “approaches” hid, by carrying only one picture.

9. The Census and the Close

What this paper measured:

1. One frequency returns exactly (Check 2.1): deviation 8.27×10^{-43} .
2. Two never do (Check 3.1): 6.38×10^{-4} out to 10^5 .
3. The constant is $\pi/\sqrt{2}$ (Check 4.1): twelve digits.
4. The good times are not sparse (Check 5.1): a largest gap of 182.182, unmoved by a fourfold range.
5. The gaps take exactly three values (Check 6.1): three even at $N = 5000$.
6. The fraction is $(\varepsilon/\pi)^d$ (Check 7.1): four digits.
7. The guess about the waiting time failed (Observation 7.3): it is ε^{1-d} .

Close: The motion made by two pendulums never returns to the same shape. And still, name any error however small and there are times at which it returns inside that error, and those times are not sparse — wait 182 and one arrives. “It does not return” and “it returns” are both true at once of one motion. The question is not which is right but in which sense it was said. And what chooses the good moments is neither a clock nor luck but the continued fraction of the frequency ratio — how far $\sqrt{2}$ stands from the rationals is exactly how well the motion can come back.

References

1. Weyl, H. (1916). Über die Gleichverteilung von Zahlen mod. Eins. *Mathematische Annalen*, 77.
2. Bohr, H. (1925). Zur Theorie der fastperiodischen Funktionen I. *Acta Mathematica*, 45.
3. Besicovitch, A. S. (1932). *Almost Periodic Functions*. Cambridge: Cambridge University Press.
4. Bohr, H. (1947). *Almost Periodic Functions*. New York: Chelsea.
5. Sós, V. T. (1958). On the distribution mod 1 of the sequence $n\alpha$. *Annales Universitatis Scientiarum Budapestinensis*, 1.
6. Khinchin, A. Ya. (1964). *Continued Fractions*. Chicago: University of Chicago Press.
7. Arnold, V. I. (1989). *Mathematical Methods of Classical Mechanics*. Second edition. New York: Springer.
8. Yamagishi, Y. (2026). *The Poincaré Recurrence Theorem and the Arrow of Time* [Paper 81]. Zenodo. 10.5281/zenodo.21733062
9. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
10. Yamagishi, Y. (2026). “Converges” Is Four Separate Claims, and What Separates Them Is What You May Discard [Paper 197]. Zenodo. 10.5281/zenodo.22107590
11. Yamagishi, Y. (2026). *What Decides Whether You Come Home Is the Exponent in the Denominator* [Paper 199]. Zenodo. 10.5281/zenodo.22108396
12. Yamagishi, Y. (2026). *Equal Volume Is Not Enough to Cut and Reassemble* [Paper 209]. Zenodo. 10.5281/zenodo.22111317
13. Yamagishi, Y. (2026). *Adding More Terms Takes You Further from the Answer* [Paper 211]. Zenodo. 10.5281/zenodo.22111772

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)