

One Sequence Goes Both to Zero and to Infinity

— In Finite Dimension Every Norm Gives the Same Answer While the Constant That Guarantees It Runs Away
with the Dimension: Approaching Is Not Settled Until a Ruler Is Chosen — [Paper 215]

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— *From thinking to syncing.* —

ABSTRACT

“It approaches” is not a property of a sequence alone. This paper sets the three rulers $\ell^1, \ell^2, \ell^\infty$ down in numbers and writes both that they are equivalent in finite dimension and that what the equivalence guarantees decays with the dimension. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the best constants between the three norms are $\sqrt{n}$ and n . At $n = 10^6$ the ℓ^1 and ℓ^∞ norms may differ by a factor of a million — and they are still “equivalent” (Sections 2 and 3). Second, that extreme lies in no ordinary direction. Among two hundred thousand random vectors the largest $\|x\|_1 / \|x\|_2$ at $n = 1000$ was only 26.21 against the bound of 31.62, which is reached only where every coordinate is equal (Section 2). Third, this is the core. For $x_n = (1/n, \dots, 1/n)$ the norm $\|x\|_1$ stays at 1 and never moves, while $\|x\|_2 = 1/\sqrt{n}$ and $\|x\|_\infty = 1/n$ both go to zero — one sequence that approaches under one ruler and does not stir under another (Section 4). Fourth, with functions it is fiercer. A spike of height $n^{1/4}$ and width $1/n$ climbs to 100 in the sup norm while falling to 0.01 in L^2 and 10^{-6} in L^1 — one sequence going both to zero and to infinity (Section 5). Fifth, at $n = 50$ the volume of the ℓ^1 unit ball is 3.04×10^{64} times smaller than that of the ℓ^∞ ball (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the equivalence of norms in finite dimension, Riesz's lemma and the volumes of the ℓ^p unit balls are all standard. The equivalence theorem is not proved — this paper measures the constants and cites the theorem. General Banach spaces are not treated — only ℓ^p and $C[0, 1]$ appear. Neither completeness nor duality is treated. No norm is called the right one — what is written is only what changes when one is chosen. Relation to Paper 177: 177 treated the order of the quantifiers and wrote the fence of compactness — the equivalence here is likewise a question of whether the constant comes before the dimension or after, the same fence from the other side. Relation to Paper 197: 197 treated the four distinct roots of “converges” — what changes here is not the kind of convergence but the way distance is measured. The corpus carries “unit ball” in twenty-five papers, and every one of them is a geometric ball; the unit ball of a normed space appears here for the first time. What is added is setting the best constants side by side at six dimensions, measuring that random samples never approach that extreme, writing out one sequence to which two norms give different answers, watching three norms split into zero, zero and infinity on a sequence of spikes, checking that against integration on a grid, and giving the ratio of unit-ball volumes at four dimensions.

1. Notation and Position

For $x \in \mathbb{R}^n$ put

$$\|x\|_1 = \sum_i |x_i|, \quad \|x\|_2 = \left(\sum_i x_i^2 \right)^{1/2}, \quad \|x\|_\infty = \max_i |x_i|.$$

Two norms are **equivalent** when $a\|x\|_A \leq \|x\|_B \leq b\|x\|_A$ holds for **some** positive a, b and **every** x . In finite dimension any two norms are equivalent.

Position: Paper 182 treated the curse of dimensionality as not one curse — what is cursed here is the constant of the equivalence. In Paper 208 curvature did the work of dimension — here dimension alone does the work.

2. Three Norms and the Best Constants

Check 2.1: the inequalities $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ always hold; going back costs the following.

n	Bound on $\ x\ _1/\ x\ _2$	Bound on $\ x\ _2/\ x\ _\infty$	Bound on $\ x\ _1/\ x\ _\infty$
2	1.414213562	1.414213562	2
10	3.16227766	3.16227766	10
100	10.0	10.0	100
1000	31.6227766	31.6227766	1000
10^6	1000.0	1000.0	10^6

Check 2.2: how rare that extreme is. Two hundred thousand Gaussian vectors at each dimension, taking the largest $\|x\|_1/\|x\|_2$.

n	Largest among 200000	$\sqrt{n}$	The all-equal direction
2	1.41421356	1.41421356	1.41421356
10	3.12127801	3.16227766	3.16227766
100	8.79139938	10.0	10.0
1000	26.20659420	31.62277660	31.62277660

Observation 2.3

A random direction does not even approach the bound. What reaches it is the single direction with every coordinate equal — one point out of the whole sphere. The worst case and the typical case are distinct roots.

3. Equivalent, and the Constant Runs Away

Observation 3.1

$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$ is true at every n , so ℓ^1 and ℓ^∞ are always equivalent in finite dimension.

Check 3.2: what that guarantee says.

n	Upper $\div$ lower	Digits they may disagree by
1	1	0.00
10	10	1.00
10^4	10^4	4.00
10^6	10^6	6.00

Observation 3.3

The theorem is true at each n , and says less the larger n is. At a million dimensions it says only that the two rulers may disagree by a factor of a million — and the word “equivalent” remains true. A word being true and a word being useful are two things.

4. The Core: One Sequence That Approaches under One Ruler and Not the Other

Check 4.1: take $x_n = (1/n, 1/n, \dots, 1/n)$ with n coordinates.

n	$\ x\ _1$	$\ x\ _2$	$\ x\ _\infty$
1	1.0	1.0	1.0
10	1.0	0.31622777	0.1
100	1.0	0.1	0.01
10^4	1.0	0.01	0.0001
10^6	1.0	0.001	10^{-6}

Observation 4.2

It is one sequence. Under ℓ^2 and ℓ^∞ it goes to zero; under ℓ^1 it does not stir. “It approaches zero” is not a property of the sequence — it is a property of the sequence together with a ruler.

Observation 4.3

The equivalence theorem is not violated here. At each n all three are equivalent — what breaks is moving n . The shape is that of Paper 211: the answer is not settled until one says which variable is being moved.

5. With Functions It Goes Both to Zero and to Infinity

On $[0, 1]$ take the spike f_n of height $n^{1/4}$ and width $1/n$.

Check 5.1:

n	Sup norm	L^2 norm	L^1 norm
10	1.7782794	0.56234133	0.17782794
100	3.1622777	0.31622777	0.031622777
10^4	10.0	0.1	0.001
10^8	100.0	0.01	10^{-6}

Observation 5.2

One sequence goes to infinity under one ruler and to zero under two others. This is no longer a question of a factor of disagreement — they disagree about whether it converges at all. And this is something only infinite dimension permits: in finite dimension it cannot happen.

6. Checked Numerically, Not Only by Formula

Check 6.1: integrate the spike on a grid of four million points and compare with the closed form.

n	Sup (grid)	L^2 (grid)	Closed form	Relative error
10	1.77827941	0.5623413252	0.5623413252	0
100	3.16227766	0.3162277660	0.3162277660	0

Observation 6.2

The grid agrees with the closed form. The previous table is not a matter of algebra but numbers obtained by actually integrating.

7. The Unit Balls, Measured

Check 7.1: the volumes of the unit balls of $\ell^1, \ell^2, \ell^\infty$.

n	ℓ^1	ℓ^2	ℓ^∞	ℓ^1/ℓ^∞
2	2.0	3.1415927	4.0	0.5
5	0.26666667	5.263789	32.0	0.0083333333
10	0.00028218695	2.550164	1024.0	2.7557319×10^{-7}
50	$3.7019019 \times 10^{-50}$	$1.7302192 \times 10^{-13}$	1.1258999×10^{15}	$3.2879494 \times 10^{-65}$

Observation 7.2

The ℓ^1 ball is $1/n!$ of the ℓ^∞ ball. At $n = 50$ that is one part in 3.04×10^{64} — two “equivalent” rulers whose unit balls differ in volume by sixty-four orders. And the ℓ^2 ball itself has shrunk to 1.73×10^{-13} by $n = 50$: the curse of dimensionality that Paper 182 measured appears here as a curse on the choice of ruler.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of “near” is which norm does the measuring. In finite dimension the choice does not change the answer — though by how much it does not worsens with the dimension. In infinite dimension the choice does change the answer — it changes whether there is convergence at all.

Observation 8.2 (the fence)

The fence around “any ruler gives the same answer” is that the dimension be finite. On the other side of that same fence stands Riesz's lemma: the closed unit ball is compact only in finite dimension. The two are one fence — the place where the constant of the equivalence runs away is the place where the ball loses its compactness.

Observation 8.3

Paper 177 wrote that the order of the quantifiers is the whole content — the equivalence here is likewise a matter of whether the constant comes before the dimension or after. Fix n and then look for a constant, and one is always found. Fix the constant first and look for one that serves every n , and there is none. One sentence turns from true to false by exchanging the order alone.

9. The Census and the Close

What this paper measured:

1. The best constants are $\sqrt{n}$ and n (Check 2.1).
2. The extreme lies in one direction only (Check 2.2): **26.21** against **31.62** among 200000.
3. Equivalent, and the constant runs away (Check 3.2): **six digits** at 10^6 .
4. One sequence does not stir under one ruler (Check 4.1): $\|x\|_1 = 1$ throughout.
5. With functions, both to zero and to infinity (Check 5.1): **100, 0.01** and 10^{-6} .
6. Grid and closed form agree (Check 6.1).
7. The unit ball is smaller by 3.04×10^{64} (Check 7.1).

Close: The sentence “this sequence approaches zero” has no meaning given the sequence alone. Measured in ℓ^1 it does not move; measured in ℓ^2 it approaches — and it is the same list of numbers. In finite dimension the difference stays a difference of factors, and exchanging rulers does not change any conclusion. But the constant that guaranteed that “does not change” runs away with the dimension — and where it has run away entirely, one sequence goes both to zero and to infinity. The theorem was not broken: we walked out of the ground it was guarding.

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