

It Approaches and Has Not Come One Step Closer

— Approaching Weakly and Approaching Strongly Are Two Things, an Orthonormal Sequence Tending Weakly to Zero While Keeping Size One: and on Squaring, the Limit Is Not Zero but Half the Average — [Paper 216]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

“It approaches” can mean that the distance shrinks or that every test stops seeing a difference. This paper separates the two in numbers and measures a sequence for which only the second happens and the first never does. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, an orthonormal sequence e_n tends to zero against every fixed test. Against $y_k = 1/k$ it falls as $1/n$ and against $y_k = 2^{-k}$ as 2^{-n} — and $\|e_n\|$ is exactly 1 throughout (Section 2). Second, no finite set of tests can ever see it. In $\mathbb{R}^{1000}$ with fifty tests one can still find a vector of length 1 on which every reading is 10^{-15} (Section 3). Third, the same holds for functions. The L^2 size of $f_n(x) = \sin(2\pi nx)$ is 0.7071067812 at every n , while the test integrals fall off as one over n (Section 4). Fourth, this is the core. Although f_n tends weakly to zero, the test integrals of f_n^2 go to 0.859140831213, which is $\frac{1}{2} \int e^x = 0.859140914230$ — the square of the weak limit is 0 and the weak limit of the square is half the average (Section 5). Fifth, in finite dimension a unit vector cannot hide. A vector of length 1 in $\mathbb{R}^N$ must read at least $1/\sqrt{N}$ on some coordinate (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — weak convergence, the Riemann–Lebesgue lemma, the Radon–Riesz property and the Banach–Alaoglu theorem are all standard. The weak topology is not constructed — what is treated is sequences, not the topology itself. The theory of compensated compactness is not developed — Section 5 touches its simplest instance only. Neither dual spaces nor reflexivity is treated. Nothing is claimed about homogenisation or turbulence closure. On a name: the corpus already carries a theorem of Riesz, in Paper 197, and that one says that convergence in measure yields an almost-everywhere convergent subsequence — the Riesz here is that weak convergence together with convergence of the norms is strong convergence, a different theorem by the same man. Relation to Paper 215: what 215 varied was which ruler does the measuring — what is varied here is what counts as a test, with the ruler held fixed. The fence is the same one, that the dimension be finite, and what fails at it is different. What is added is setting the test values and the size of an orthonormal sequence side by side, actually constructing a vector invisible to finitely many tests, measuring that the size of an oscillating sequence does not move while its test integrals do, confirming with three test functions that the weak limit of the square is half the average, measuring that the difference vanishes once the norms agree, and confirming by search that the floor in finite dimension is $1/\sqrt{N}$.

1. Notation and Position

For a sequence $\mathbf{x}_n$ in an inner-product space, **approaching strongly** means $\|\mathbf{x}_n - \mathbf{x}\| \rightarrow 0$, and **approaching weakly** means $\langle \mathbf{x}_n, \mathbf{y} \rangle \rightarrow \langle \mathbf{x}, \mathbf{y} \rangle$ for every fixed $\mathbf{y}$. Strong implies weak, and the converse fails.

Position: Paper 197 separated the four roots of “converges” by what one is allowed to discard — what is separated here is what one goes to look at. In Paper 215 “near” depended on the choice of ruler — here the ruler is one and the ways of looking are two.

2. An Orthonormal Sequence: Tests to Zero, Size Fixed at One

Check 2.1: let $\mathbf{e}_n$ be the n -th coordinate vector of ℓ^2 and test it against two fixed vectors.

n	Test against $y_k = 1/k$	Test against $y_k = 2^{-k}$	$\ e_n\ $
1	1.0	0.5	1.0
5	0.2	0.03125	1.0
20	0.05	9.5367432×10^{-7}	1.0
100	0.01	$7.8886091 \times 10^{-31}$	1.0
1000	0.001	$9.3326362 \times 10^{-302}$	1.0

Observation 2.2

Every fixed test goes to zero and the size never moves once. $\|e_n - 0\| = 1$ for ever — “it is approaching zero” and “it stays at distance one from zero” are both true of the same sequence.

3. No Finite Set of Tests Can See It

Check 3.1: in $\mathbb{R}^n$ take k random tests and look for a vector of length 1 on which every test reads zero.

n	k	Largest reading	Length of the vector
10	5	6.661×10^{-16}	1.000000000000
100	5	3.886×10^{-16}	1.000000000000
1000	5	6.661×10^{-16}	1.000000000000
1000	50	1.887×10^{-15}	1.000000000000

Observation 3.2

So long as the dimension exceeds the number of tests some direction is invisible to them. Weak convergence is that loophole taken to the limit — the tests may be infinite in number, but they are used one fixed at a time, so there is always a direction not yet looked at.

4. The Same with Functions: Faster and Faster Oscillation

Check 4.1: take $f_n(x) = \sin(2\pi nx)$ on $[0, 1]$.

n	$\ f_n\ _{L^2}$	$\int f_n x$	$\int f_n e^x$	$\int f_n/(1+x)$
1	0.7071067812	-1.592×10^{-1}	-2.667×10^{-1}	7.401×10^{-2}
16	0.7071067812	-9.947×10^{-3}	-1.709×10^{-2}	4.972×10^{-3}
64	0.7071067812	-2.487×10^{-3}	-4.273×10^{-3}	1.243×10^{-3}
256	0.7071067812	-6.217×10^{-4}	-1.068×10^{-3}	3.108×10^{-4}

($1/\sqrt{2} = 0.7071067812$; the integrals use a grid of 2^{22} points.)

Observation 4.2

The test integrals fall as one over n (Riemann–Lebesgue) and the size does not move to ten decimal places. The wave only gets faster and does not get smaller — what is going to zero is not the wave but its engagement with a smooth partner.

5. The Core: Squaring Moves the Limit

Check 5.1: test f_n^2 for the same f_n .

n	$\int f_n^2 x$	$\int f_n^2 e^x$	$\int f_n^2/(1+x)$
1	0.250000000000	0.853734576864	0.344299547225
16	0.250000000000	0.859119662515	0.346564315769
256	0.250000000000	0.859140831213	0.346573554045
$\frac{1}{2} \int g$	0.250000000000	0.859140914230	0.346573590280
Square of the weak limit	0	0	0

Observation 5.2

The weak limit of the square is not the square of the weak limit. $f_n \rightarrow 0$ weakly and $f_n^2 \rightarrow \frac{1}{2}$ weakly — not 0 but half the average. All three test functions agree with $\frac{1}{2} \int g$ to seven decimal places.

Observation 5.3

Here is the price of approaching weakly. Strong convergence carries squares along with it; weak convergence passes only what is linear — and the square is the energy, the variance, the intensity. So only quantities of first order may be called gone, and those of second order remain.

Observation 5.4

One can even say where what seemed to vanish has gone — the oscillation merely got faster, and the energy is constant. This paper does not go on from here to homogenisation or turbulence closure: only the simplest instance has been touched.

6. Once the Norms Agree, Weak Becomes Strong

Check 6.1: $x_n = (1 - 1/n)e_1 + (1/n)e_2$ tends weakly to e_1 .

n	$\ x_n\ $	$\ e_1\ $	$\ x_n - e_1\ $
2	0.707106781187	1.0	0.707106781187
10	0.905538513814	1.0	0.141421356237
100	0.990050503762	1.0	0.014142135624
10000	0.999900005001	1.0	0.000141421356

Observation 6.2

Once the sizes settle onto the size of the limit, the difference goes to zero (Riesz). For the orthonormal sequence of Section 2 the size stays at 1 and does not settle onto the weak limit's size of 0 — so the difference does not go to zero either. Adding one single thing to weak convergence makes it strong, and that thing is that the sizes agree.

7. In Finite Dimension a Unit Vector Cannot Hide

Check 7.1: put a vector of length 1 in $\mathbb{R}^N$ through the N coordinate tests. How small can **the largest reading** be made?

N	Least possible, $1/\sqrt{N}$	Least among 4000 random
1	1.0	1.0
4	0.5	0.51812550
100	0.1	0.19566903
10000	0.01	0.03278663

Observation 7.2

The floor is $1/\sqrt{N}$, and it is positive at every finite N . A vector of length 1 must show itself on some test — the floor reaches zero only where N does not stop, and that is where weak and strong part. Random directions do not reach the floor (0.0328 against 0.01 at $N = 10^4$): it is reached only by the all-equal direction.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the two senses of approaching is what one goes to look at.

Phrase	Demand	For e_n
Approaches strongly	the distance goes to zero	does not happen (stays 1)
Approaches weakly	every fixed test loses the difference	happens
Weakly, and the sizes agree	both of the above	does not happen

Observation 8.2 (the fence)

The fence around “approaching means getting nearer” is that the dimension be finite. It is the same fence as in Paper 215 with a different thing failing at it: there it was which ruler is chosen that began to matter, and here it is that finitely many tests no longer suffice. Outside one fence, several distinct failures stand side by side.

Observation 8.3

And one more. Under weak convergence only linear quantities are carried to the limit — and the moment one squares, 0 turns into $\frac{1}{2}$. That taking the limit and applying a function cannot be exchanged is exactly the price of having left strong convergence. In the language of Paper 177 this too was a question of order.

9. The Census and the Close

What this paper measured:

1. Tests to zero, size fixed at one (Check 2.1).
2. Invisible to finitely many tests (Check 3.1): length 1 even with fifty tests in a thousand dimensions.
3. Oscillation does not shrink (Check 4.1): fixed at 0.7071067812.
4. The weak limit of the square is half the average (Check 5.1): 0.859140831213 against 0.859140914230.
5. Once the sizes agree the difference goes (Check 6.1): 1.41×10^{-4} .
6. The floor in finite dimension is $1/\sqrt{N}$ (Check 7.1).

Close: The sequence $e_1, e_2, e_3, \dots$ is approaching zero. Take its inner product with any fixed partner and the value falls away to nothing. And yet its distance from zero is 1 from first to last — it has not come one step closer. That it could be called approaching came of looking only at tests of first order: the moment one looks at the square, a limit

that ought to have been 0 is half the average. What vanished was not the wave but only the wave's engagement with a smooth partner. The wave is still there, and has merely become too fast for the tests we brought.

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Yuuki YAMAGISHI (山岸 勇輝)

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