

Ask Only That Distances Be Kept, and Straightness Is Kept Too

— A Surjective Isometry Must Be Affine, and the Moment Room Is Left in the Target It May Bend by Any

Function of Slope at Most One — [Paper 217]

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— *From thinking to syncing.* —

ABSTRACT

An isometry is asked one thing only, that distances be kept, and nothing whatever about addition or scaling. And still, provided only that it is onto, it must be affine (Mazur–Ulam). This paper measures how much work that “provided only” is doing. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, a surjective isometry really is affine. Testing a rotation with a shift on two hundred thousand random pairs, the error in the distances is 5.33×10^{-15} and the bend at the midpoint 1.78×10^{-15} (Section 2). Second, this is the core. Mapping the line into the plane with the max-norm by $t \mapsto (t, g(t))$ keeps every distance exactly, for any g whatever of slope at most one — the error in the distances is exactly zero and the bend at the midpoint reaches **19.94** for $g = |t|$ (Section 3). Third, the same map is not an isometry for the ordinary norm. Measured in ℓ^2 the ratio rises to **1.41421356** — what the counterexample leans on is the choice of norm (Section 4). Fourth, what does the work is whether the midpoint is one point. Tightening the tolerance from 10^{-2} to 10^{-4} shrinks the diameter of the midpoint set at every finite p , and for the max-norm it does not move: $2.0180 \rightarrow 2.0020 \rightarrow \mathbf{2.0000}$ (Section 5). Fifth, over the complex numbers only real straightness follows. Conjugation keeps every distance with error 0, and $|\overline{iz} - iz|$ reaches **8.478** (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the Mazur–Ulam theorem, the existence of non-surjective isometric embeddings and uniform convexity are all standard. The theorem is not proved — this paper measures the force of the fence and cites the theorem. The general theory of the non-surjective case (Figiel, Godefroy–Kalton) is not developed — what is given is one counterexample. Isometries of infinite-dimensional spaces are not treated — every number here belongs to the line and the plane. Lipschitz-free spaces are not touched. Relation to Paper 212: that paper wrote that no isometry from the sphere to the plane exists — this one writes an isometry that exists and is bent. The same word, failing in a different way. The corpus carries “isometry” in two papers, 11 and 41, in passing only; Mazur–Ulam appears here for the first time. What is added is confirming the affineness of a surjective isometry on two hundred thousand random pairs, actually building the counterexample with three functions of slope at most one and measuring the bend, measuring that the same map is not an isometry for another norm, measuring the diameter of the midpoint set while tightening the tolerance, measuring that this softness grows steadily with p , and measuring that conjugation keeps distances without being complex-linear.

1. Notation and Position

A map F is an **isometry** when $\|F(x) - F(y)\| = \|x - y\|$ for all x, y . Neither addition nor scaling appears in that. It is **affine** when $F(x) - F(0)$ is linear. The Mazur–Ulam theorem says that a **surjective** isometry between real normed spaces is affine.

Position: In Paper 212 the non-existence of an isometry from the sphere to the plane was written in terms of curvature — here an isometry that does exist is not affine. In Papers 215 and 216 the fence was that the dimension be finite — the fence here is that the map be onto.

2. If It Is Onto, It Really Is Affine

Check 2.1: a rotation of the plane through **0.7** with a shift of **(0.3, −1.2)**, on two hundred thousand random pairs.

Measured	Largest
Error in the distances	5.329×10^{-15}
Bend at the midpoint	1.776×10^{-15}

Observation 2.2

Linearity was not assumed. Two things were imposed, that distances be kept and that every point be reached, and the midpoint identity held as a consequence, to the last digit.

3. The Core: Without Surjectivity It Bends as Much as You Like

Map the line into the plane with the **max-norm** by $t \mapsto (t, g(t))$. If g has slope at most one then

$$\|(s, g(s)) - (t, g(t))\|_\infty = \max(|s - t|, |g(s) - g(t)|) = |s - t|,$$

so this is an isometry.

Check 3.1: four hundred thousand random pairs s, t drawn from $[-20, 20]$.

g	Largest error in the distances	Largest bend at the midpoint
$g(t) = \sin t$	0	1.999991
$g(t) = t $	0	19.942160
$g(t) = \text{a sawtooth}$	0	0.999193

Observation 3.2

The left column is exactly zero: these really are isometries. The right is not zero: they are not affine. And g may be anything at all of slope at most one — the way it bends is not determined, and carries a whole function's worth of freedom.

Observation 3.3

Not being onto is the only difference here. The image is one curve inside the plane and not the plane — the moment room is left, the theorem says nothing at all.

4. The Same Map Is Not an Isometry for the Ordinary Norm

Check 4.1: measure the same pairs in ℓ^2 .

g	Largest $\ F(s) - F(t)\ _2/ s - t $
$g(t) = \sin t$	1.41420840
$g(t) = t $	1.41421356
$g(t)$ = a sawtooth	1.41421356

Observation 4.2

For an isometry this would have to be one, and it rises to $\sqrt{2}$. What the counterexample leans on is not the ingenuity of the map but the choice of norm.

5. What Does the Work: Is the Midpoint One Point?

The proof of Mazur–Ulam uses **that the midpoint is preserved**. If the midpoint is **determined by distances alone**, an isometry cannot move it.

Check 5.1: for $\boldsymbol{x} = (-1, 0)$ and $\boldsymbol{y} = (1, 0)$, collect the $\boldsymbol{z}$ with $\|\boldsymbol{z} - \boldsymbol{x}\| = \|\boldsymbol{z} - \boldsymbol{y}\| = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|$ to a tolerance, and measure the diameter.

p	Tolerance 10^{-2}	10^{-3}	10^{-4}	Limit
1	0.0180	0.0020	0.0000	0 (one point)
2	0.2820	0.0880	0.0280	0 (one point)
3	0.6220	0.2880	0.1320	0 (one point)
10	1.5940	1.2620	1.0020	0 (one point)
∞	2.0180	2.0020	2.0000	2 (a segment)

Observation 5.2

At every finite p the diameter shrinks as the tolerance tightens — the midpoint is one point and there is a peg to pin the map to. For the max-norm it does not shrink at all — the midpoint is a segment of length two, and there is no peg.

Observation 5.3

The theorem itself does not require uniqueness of the midpoint — even in the max-norm, if the map is onto it is affine. What uniqueness governs is how far it may bend once surjectivity is dropped.

6. The Softening Is Gradual

Check 6.1: the diameter of the set within $\varepsilon = 0.01$ of being a midpoint.

p	Diameter
1.5	0.1200
2	0.2820
3	0.6220
6	1.2560
10	1.5940
30	1.9300

Observation 6.2

The room available grows smoothly with p . “Is the midpoint one point” looks like a yes or no and is in practice a quantity — by $p = 30$ it is nearly the max-norm already. The truth of the theorem switches at $p = \infty$, and its force fades continuously.

7. Over the Complex Numbers, Only Real Straightness Follows

Check 7.1: complex conjugation $z \mapsto \bar{z}$, on two hundred thousand pairs.

Measured	Largest
Error in the distances	0
Departure from complex linearity, $ \overline{iz} - i\bar{z} $	8.478225

Observation 7.2

Conjugation is onto, keeps every distance exactly, and is linear in the real sense. And it is not linear in the complex sense — what distance forces is straightness over the reals and no more. “Linear” too is unsettled until one says over which field.

8. The Separator and the Fence

Observation 8.1 (the separator)

What decides whether an isometry is affine is whether it fills the target.

Condition	Example	Affine?
Onto, real normed spaces	rotation with a shift	necessarily
Not onto	$t \mapsto (t, \sin t)$ in the max-norm	no
Onto, complex linearity demanded	conjugation	real yes, complex no

Observation 8.2 (the fence)

The fence around “keeping distances keeps straightness” is that the map be onto. Outside the fence the sentence is false — and not slightly false: it may bend freely by a whole function of slope at most one.

Observation 8.3

Set this beside Paper 212. There, an isometry from the sphere to the plane did not exist — here, isometries from the line to the plane exist, and there are infinitely many. “Whether an isometry exists” and “whether an isometry is well behaved” were two different questions. In Papers 215 and 216 one fence had two different failures beyond it — here again distinct failures stand under one word.

9. The Census and the Close

What this paper measured:

1. **Onto implies affine** (Check 2.1): **bend at the midpoint 1.78×10^{-15} .**
2. **Not onto permits bending** (Check 3.1): **distance error 0, bend 19.94.**
3. **Not an isometry for another norm** (Check 4.1): **the ratio rises to $\sqrt{2}$.**
4. **One point at finite p , a segment at ∞** (Check 5.1): **2.0000, unmoved.**
5. **The softening is gradual** (Check 6.1): **1.9300 at $p = 30$.**
6. **Conjugation is linear only over the reals** (Check 7.1): **departure 8.478.**

Close: One thing only was asked of an isometry: keep the distances. Nothing was said of addition, nothing of scaling. And still, so long as it fills its target, such a map must be affine — a condition that only measures compels a shape that is algebraic. Yet leave the smallest room in the target and the same condition says almost nothing — choose any function of slope at most one and bend by exactly that much. What made the theorem work was not distance but having nowhere to escape to.

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