

There Are Four Independences, and They Say Nothing to One Another

— Being Uncorrelated Demands One Moment and Being Independent Demands Them All: Fourteen of Twenty

Hold of Their Own Accord, and the One Everybody Checks Is Among Them — [Paper 220]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

The word “independent” carries a different meaning in linear algebra, in probability and in field theory. This paper sets four of them in one table and writes in numbers that any one of them holding says nothing about the others. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the failure of independence is sparse. Taking X uniform on $[-1, 1]$ and $Y = X^2$, which is as dependent as anything can be, and comparing $E[X^a Y^b]$ with $E[X^a]E[Y^b]$ over twenty pairs, fourteen hold and only six fail (Section 2). Second, the one pair everybody checks is among those that hold. The gap at $(1, 1)$, which is the correlation, is exactly zero, and the failures begin at $(2, 1)$ with 0.08888888889 (Section 2). Third, this is the core. With X, Y fair coins and $Z = X \oplus Y$, all three pairs have a gap of zero while the three together are off by -0.125 — and in four million samples $P(X = Y = Z = 1) = 0.00000000$ against $1/8 = 0.125$ (Section 4). Fourth, linear and statistical independence cannot see one another. The rank of the pair of sample vectors was **2** both for X with X^2 and for two genuinely independent variables (Section 5). Fifth, algebraic independence cannot be settled by search. For (π, π^2) coefficients up to two already find the exact relation $y = x^2$, and for (π, e) the least value reached is 0.00017823786 and did not improve thereafter (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the gap between pairwise and mutual independence (Bernstein's example), transcendence theory and linear independence are all standard. The algebraic independence of π and e is not decided — it is an open problem, and all this paper does is rule out small relations. No test of independence is proposed. Measuring dependence by information (mutual information) is not treated. Categorical independence and measure-theoretic products are not treated.

Relation to Paper 187: 187 showed, inside probability, that being uncorrelated and being independent are different, confirming it with the sample mean and variance — what is set out here is four notions from outside probability, and nothing new is claimed inside it. What is added is turning that single observation of 187 into a map, a grid of moments. What is added is counting the failures over a grid of twenty pairs and finding fourteen against six, checking that against four million samples, giving an example, exactly and by sampling, of three things independent in pairs and not together, showing by rank that linear independence cannot tell the two cases apart, and running a search for algebraic relations while raising the bound on the coefficients.

1. Notation and Position

Four senses of “independent”. **Linearly independent**: no nontrivial **linear combination** vanishes. **Algebraically independent**: no nontrivial **polynomial with integer coefficients** vanishes. **Statistically independent**: the joint distribution is **the product of the margins**. **Uncorrelated**: the single equation $E[XY] = E[X]E[Y]$ holds.

Position: Paper 197 divided “converges” into four, separated by what one is allowed to discard — what separates them here is what one demands shall not vanish. Paper 187 already showed inside probability that uncorrelated and independent differ — this paper adds a map and steps outside.

2. The Failure Is Sparse

Take X uniform on $[-1, 1]$ and $Y = X^2$. Since $E[X^k]$ is zero for odd k and $1/(k+1)$ for even k , everything is exact.

Check 2.1: the twenty pairs with $a = 0, \dots, 4$ and $b = 0, \dots, 3$.

a	b	$E[X^a Y^b]$	$E[X^a]E[Y^b]$	Gap
1	1	0.0	0.0	0.0
1	2	0.0	0.0	0.0
3	1	0.0	0.0	0.0
2	1	0.2	0.1111111111	0.08888888889
2	2	0.1428571429	0.06666666667	0.07619047619
4	1	0.1428571429	0.06666666667	0.07619047619

Of the twenty, **14** hold and **6** fail.

Observation 2.2

The gap at $(1, 1)$, which is the correlation, is exactly zero. Every pair with a odd holds automatically by symmetry, and the failures sit only in the corner where a is even and b is not zero — the failure is sparse, and the one pair everybody checks is on the side that holds.

Observation 2.3

To say “it is uncorrelated, so there is no evidence against independence” is to have looked at one of twenty pairs and not at the other nineteen. And that one pair sits in the place hardest to break.

3. Checked against Samples

Check 3.1: the same gaps in four million samples.

(a, b)	Gap in the sample	Exact	Difference
(1, 1)	0.0001688246	0.0000000000	1.69×10^{-4}
(2, 1)	0.0889025568	0.0888888889	1.37×10^{-5}
(2, 2)	0.0762082743	0.0761904762	1.78×10^{-5}
(4, 1)	0.0762082743	0.0761904762	1.78×10^{-5}

Observation 3.2

The samples agree with the exact values. But the sample value at (1, 1) is 1.7×10^{-4} and not zero — with four million points, reading a zero as a zero still requires knowing the size of the error.

4. The Core: Independent in Pairs and Not Together

Let X, Y be fair coins and $Z = X \oplus Y$, the exclusive or.

Check 4.1:

Group	$P(\text{both} = 1)$	Product of margins	Difference
X and Y	0.25	0.25	0.0
X and Z	0.25	0.25	0.0
Y and Z	0.25	0.25	0.0
All three	0	0.125	-0.125

(In four million samples $P(X = Y = Z = 1) = 0.00000000$.)

Observation 4.2

Any two of them are perfectly independent, and the three together are perfectly dependent. Fix two and the third is settled outright — however many times “independent in pairs” is confirmed, it comes not one step nearer to independence of the three.

Observation 4.3

This has the shape of Section 2: independence is a conjunction of many conditions, and checking part of it says nothing about the rest. And the conditions easiest to check are the likeliest to hold.

5. Linear and Statistical Independence Cannot See One Another

Check 5.1: take $n = 6$ samples and stack the list of $\mathbf{X}$ and the list of $\mathbf{Y} = \mathbf{X}^2$ as two vectors in $\mathbb{R}^6$.

Pair	Rank	Statistically independent?
$\mathbf{X}$ and $\mathbf{X}^2$	2	no ($\mathbf{Y}$ is a function of $\mathbf{X}$)
Two independent variables	2	yes

Observation 5.2

Both have rank two, that is, both are linearly independent. Linear independence cannot tell the two cases apart — and statistical independence does not look at rank at all. These two are not rungs of one ladder but different axes.

6. Algebraic Independence: What a Search Can and Cannot Do

Look for integers p_{ij} , not all zero, with $\sum p_{ij}x^i y^j = 0$, over the six terms with $i + j \leq 2$, raising the bound on the coefficients.

Check 6.1: for (π, π^2) a search with coefficients up to two gives a **least value of 0.0**, that is, an **exact relation**: $-2y + 2x^2 = 0$.

Check 6.2: for (π, e) .

Bound on the coefficients	Candidates	Least value found
2	15625	0.002700226
3	117649	0.00017823786
4	531441	0.00017823786
5	1771561	0.00017823786

Observation 6.3

For (π, π^2) the relation appears at once, at the smallest coefficients. For (π, e) nothing exact appears; the least value fell by a factor of fifteen from bound two to bound three and did not improve at four or five.

Observation 6.4

Neither the falling nor the stopping is evidence of anything. Allow the coefficients to grow far enough and such a sum can be brought arbitrarily near zero for any two real numbers whatever — so a small value proves nothing. A search can rule out small relations and can never settle the question. Whether π and e are algebraically independent is open (Nesterenko proved in 1996 that π and e^π are independent).

7. The Four Demands, Side by Side

Census 7.1:

Kind	What must not vanish	How many conditions
Linearly independent	any nontrivial linear combination	one per combination
Algebraically independent	any nontrivial polynomial	infinitely many
Statistically independent	every moment factorises	infinitely many
Uncorrelated	the (1, 1) moment factorises	one

Observation 7.2

The first two are about exact relations among fixed numbers or vectors; the last two are about whether a distribution splits into a product. They are different axes, and a fact on one says nothing on the other.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the four is how wide a family of combinations is required not to vanish — linear combinations only, all polynomials, all moments, or one moment. The wider the demand the stronger the condition, and the harder to check.

Observation 8.2 (the fence)

The fence around “independence has been checked” is which family it was checked over. Outside that fence the phrase is not even false but empty — in Section 2 one of twenty pairs was looked at, and in Section 4 only the three pairs. Both could be called “checked”, and both checked nothing.

Observation 8.3

And the condition easiest to check is the hardest to break. The correlation sits among the fourteen that hold, and all three pairwise tests passed perfectly. A test being passed easily is not because the object is well behaved but because the test is weak. In Paper 216 weak convergence passed only tests of first order — here too, a weak independence looks at one moment only.

9. The Census and the Close

What this paper measured:

1. The failure is sparse (Check 2.1): **fourteen of twenty hold.**
2. The pair everybody checks is among them (Check 2.1): **the gap at (1, 1) is zero.**
3. Samples agree (Check 3.1): **differences of order 10^{-5} .**

4. Independent in pairs, dependent in three (Check 4.1): **−0.125**.
5. Linear independence cannot tell them apart (Check 5.1): **rank two both times**.
6. A search cannot settle the question (Check 6.2): **stopping at 0.00017823786**.

Close: Nothing is more dependent than X and X^2 — fix one and the other is fixed entirely. And still the correlation is exactly zero, and fourteen of twenty moment conditions hold of their own accord. Set three coins side by side and any two of them are perfectly independent while the three together leave no freedom at all. To be entitled to say “independence was checked” one must say over which family it was checked — and the families easiest to name are the easiest to pass. That a test was passed is often not news about the object but news about the test.

References

1. Bernstein, S. N. (1946). *Theory of Probability*. Moscow: Gostekhizdat.
2. Halmos, P. R. (1958). *Finite-Dimensional Vector Spaces*. Second edition. Princeton: Van Nostrand.
3. Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Volume 1. Third edition. New York: Wiley.
4. Baker, A. (1975). *Transcendental Number Theory*. Cambridge: Cambridge University Press.
5. Nesterenko, Yu. V. (1996). Modular functions and transcendence questions. *Sbornik: Mathematics*, 187.
6. Lang, S. (2002). *Algebra*. Third edition. New York: Springer.
7. Yamagishi, Y. (2026). *The Gaussian Is Special for More Than One Reason* [Paper 187]. Zenodo. 10.5281/zenodo.22105639
8. Yamagishi, Y. (2026). *“Converges” Is Four Separate Claims, and What Separates Them Is What You May Discard* [Paper 197]. Zenodo. 10.5281/zenodo.22107590
9. Yamagishi, Y. (2026). *One Sequence Goes Both to Zero and to Infinity* [Paper 215]. Zenodo. 10.5281/zenodo.22119477
10. Yamagishi, Y. (2026). *It Approaches and Has Not Come One Step Closer* [Paper 216]. Zenodo. 10.5281/zenodo.22119579
11. Yamagishi, Y. (2026). *Ask Only That Distances Be Kept, and Straightness Is Kept Too* [Paper 217]. Zenodo. 10.5281/zenodo.22119628
12. Yamagishi, Y. (2026). *A Flat Slab Brings Light to a Focus* [Paper 219]. Zenodo. 10.5281/zenodo.22119788

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)