

Change the Last Bit and an Eigenvalue Moves by 0.16

— Being Diagonalisable Looks Like a Yes or No and Is in Fact the n -th Root of ε : in a Jordan Block of Order Twenty, Rounding Alone Scatters the Eigenvalues Around a Circle — [Paper 221]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

“Is it diagonalisable” is a question answered yes or no. This paper writes in numbers what that answer fails to guarantee. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, the law is $\varepsilon^{1/n}$. Putting 10^{-16} in the corner of an $n \times n$ Jordan block gives eigenvalues of size 10^{-4} at $n = 4$, **0.0251188643151** at $n = 10$ and **0.158489319247** at $n = 20$ — with a ratio to the prediction of 1.0000000000 in every case (Section 2). Second, the eigenvalues lie evenly around a circle. At $n = 12$ every radius is **0.0464158883361** and every angular gap is **0.52359878**, which is $2\pi/12$ (Section 3). Third, this is the core. Under a random disturbance of the same size, the Jordan block moves by **0.14973076** at $n = 20$ while a well-separated diagonal matrix moves by 6.22×10^{-15} — a factor of 10^{15} for the same disturbance (Section 4). Fourth, the yes or no is in truth a quantity. A 2×2 matrix whose eigenvalues differ by δ is diagonalisable at every δ , and the condition number of its eigenvectors reaches 2×10^9 by $\delta = 10^{-9}$ (Section 5). Fifth, a Jordan block never appears in floating point. Adding a random disturbance of size 10^{-16} gave n distinct eigenvalues in all two hundred trials, at $n = 5, 10$ and **20** alike (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the Jordan form, Wilkinson's perturbation analysis of eigenvalues and pseudospectra are all standard. No numerical method is proposed — this paper measures with existing tools. The Schur decomposition is not computed — that a stable computation exists is cited and no more. The general theory of non-normal matrices is not developed — what appears is the Jordan block and one family of 2×2 matrices. Nothing is claimed about any application. Relation to Paper 219: there, a lens free of aberration existed at a single point in the space of materials — the Jordan block here is likewise a single point in the space of matrices, and moreover arithmetic can never stand on it. Relation to Paper 217: there, the truth of a theorem switched at $p = \infty$ while its force faded continuously — the same thing happens here. What is added is confirming the law at six dimensions with a ratio of 1.0000000000, measuring that the eigenvalues are evenly spaced on a circle, comparing the movement under the same disturbance with a diagonal matrix and obtaining a factor of 10^{15} , measuring the condition number and the movement against δ , finding the crossing of the two regimes at $\varepsilon = \delta^2$, counting that a Jordan block appears in none of two hundred trials, and confirming that the radius of the pseudospectrum equals $\varepsilon^{1/n}$.

1. Notation and Position

An $n \times n$ Jordan block has ones just above the diagonal and zeros elsewhere. Its only eigenvalue is 0, with multiplicity n , and it has only one eigenvector. Throughout, $\varepsilon = 10^{-16}$, about the gap between 1 and its neighbour in double precision.

Position: In Paper 217 the uniqueness of the midpoint switched at $p = \infty$ while the softness was gradual — in Paper 219 the perfect lens existed at a single point. This paper is where those two happen at once.

2. The Law: It Moves by the n -th Root of ε

Check 2.1: put $\varepsilon = 10^{-16}$ in the bottom-left corner of the block.

n	Largest eigenvalue	$\varepsilon^{1/n}$	Ratio
2	10^{-8}	10^{-8}	1.0000000000
4	10^{-4}	10^{-4}	1.0000000000
8	0.01	0.01	1.0000000000
10	0.0251188643151	0.0251188643151	1.0000000000
16	0.1	0.1	1.0000000000
20	0.158489319247	0.158489319246	1.0000000000

Observation 2.2

Altering the sixteenth digit alone moves the eigenvalues of a twenty by twenty matrix by **0.16**. That is 10^{15} times ε itself — and the sentence “matrices that are close have close eigenvalues” does not hold here.

3. And They Scatter Evenly Around a Circle

Check 3.1: the radii and arguments at $n = 12$.

Measured	Value
Radii, least to greatest	0.0464158883361 to 0.0464158883361
Angular gaps, least and greatest	0.52359878 and 0.52359878
$2\pi/12$	0.52359878

Observation 3.2

The twelve eigenvalues are the twelfth roots of ε , spaced perfectly evenly around a circle. One eigenvalue of multiplicity n has turned into n simple ones — and not one of them is real: the original matrix has the single eigenvalue 0.

4. The Core: Under the Same Disturbance a Diagonalisable Matrix Moves 10^{15} Times Less

Check 4.1: twenty **random** disturbances of norm ε applied to each.

Matrix	n	Largest movement	Movement $\div \varepsilon$
Jordan block	10	0.02216382	2.22×10^{14}
Diagonal, well separated	10	4.44×10^{-15}	44.4
Jordan block	20	0.14973076	1.50×10^{15}
Diagonal, well separated	20	6.22×10^{-15}	62.2

($\varepsilon^{1/n}$ at $n = 20$ is 0.15848932.)

Observation 4.2

Under disturbances of exactly the same size, one moves by 6×10^{-15} and the other by 0.15 — a factor of 10^{15} . The size of the matrix and the size of the disturbance are the same in both; the only difference is whether the eigenvectors form a basis.

5. The Yes or No Is in Truth a Quantity

Let $A(\delta)$ be the 2×2 matrix whose $(1, 2)$ entry is 1 and whose $(2, 2)$ entry is δ , the other two being 0. Its eigenvalues are 0 and δ , so it is **diagonalisable** for every $\delta \neq 0$.

Check 5.1:

δ	Condition number of the eigenvectors	Movement under ε
10^{-1}	20.0499	9.99×10^{-16}
10^{-3}	2000	10^{-13}
10^{-5}	200000	10^{-11}
10^{-7}	2×10^7	9.90×10^{-10}
10^{-9}	2×10^9	9.51×10^{-9}

Observation 5.2

“Diagonalisable” is true at every δ , and the basis becomes arbitrarily bad. The word is yes or no and what it stands for is continuous — what switches at $\delta = 0$ is only the word, and nothing switches in the numbers.

6. Two Regimes, and Where They Cross

Check 6.1: for the same $A(\delta)$, vary the disturbance.

δ	ε	Movement	ε/δ	$\sqrt{\varepsilon}$
10^{-3}	10^{-14}	10^{-11}	10^{-11}	10^{-7}
10^{-3}	10^{-8}	9.90×10^{-6}	10^{-5}	10^{-4}
10^{-3}	10^{-6}	6.18×10^{-4}	10^{-3}	10^{-3}
10^{-5}	10^{-14}	10^{-9}	10^{-9}	10^{-7}
10^{-5}	10^{-10}	6.18×10^{-6}	10^{-5}	10^{-5}

Observation 6.2

While the disturbance is small the movement follows ε/δ , and once ε passes δ^2 it changes over to $\sqrt{\varepsilon}$. The crossing is at $\varepsilon = \delta^2$ — and the change-over is smooth: there is no moment at which the matrix stops being diagonalisable. It has the shape of what happened in Paper 217 as p was raised.

7. A Jordan Block Never Appears in Floating Point

Check 7.1: add a random matrix of size ϵ to the block and count the distinct eigenvalues, two hundred times each.

n	Distinct eigenvalues	Out of
5	always 5	200/200
10	always 10	200/200
20	always 20	200/200

Observation 7.2

Not once does a repetition survive. Matrices with a repeated eigenvalue form a set of measure zero, and rounding always pushes one outside it — so “this matrix is a Jordan block” can never be confirmed by computation. As in Paper 219 the perfect lens stood at a single point in the space of materials, so here a single point in the space of matrices, and arithmetic cannot stand on it.

8. What Is Honest Instead: The Pseudospectrum, and the Separator and the Fence

Instead of the eigenvalues, take the set of z where the smallest singular value of $zI - J$ falls below ϵ : the **pseudospectrum**.

Check 8.1: its radius.

n	Radius of the pseudospectrum	$\epsilon^{1/n}$	Ratio
4	0.0001000000003	0.0001	1.000000
8	0.01000012501	0.01	1.000013
12	0.04642423441	0.04641588834	1.000180
16	0.1000629138	0.1	1.000629

Observation 8.2

The radius of the pseudospectrum is the radius to which the eigenvalues actually moved. The spectrum says “one point” and the pseudospectrum says “a disc” — and what arithmetic sees is the disc.

Observation 8.3 (the separator)

What separates the senses of “eigenvalue” is which matrix is being spoken of — that matrix exactly, or everything near it. The first gives the single point 0 and the second a disc of radius $\varepsilon^{1/n}$, and both are correct.

Observation 8.4 (the fence)

The fence around “matrices that are close have close eigenvalues” is that the eigenvectors form a basis. Outside it the sentence is false, and false by a factor of 10^{15} . What happened in Paper 220 happens here: the test “is it diagonalisable” is passed, and passing it guarantees nothing — what would guarantee something is the condition number, which appears nowhere in the answer.

9. The Census and the Close

What this paper measured:

1. The law is $\varepsilon^{1/n}$ (Check 2.1): a ratio of 1.0000000000 at all six dimensions.
2. Evenly spaced on a circle (Check 3.1): $2\pi/12 = 0.52359878$.
3. A factor of 10^{15} under the same disturbance (Check 4.1): 0.14973076 against 6.22×10^{-15} .
4. The yes or no is a quantity (Check 5.1): a condition number of 2×10^9 .
5. The crossing is at $\varepsilon = \delta^2$ (Check 6.1): and the change-over is smooth.
6. The Jordan block never appears (Check 7.1): n distinct in 200/200.
7. The pseudospectrum gives the true radius (Check 8.1): six digits at $n = 4$.

Close: A Jordan block of order twenty has the single eigenvalue 0 — and that is perfectly correct. Change one last bit and those eigenvalues scatter around a circle of radius 0.16 — and that is perfectly correct too. The two do not conflict: the first is a statement about that matrix exactly and the second about every matrix near it. The trouble is that we never know which of the two a measured matrix is. The answer “it is diagonalisable” contained nothing at

all of the meaning “it will survive computation” — that meaning is carried by the condition number, which is written nowhere in the answer.

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