

Flat Wherever You Touch It, and Four Pi in Total

— Curvature Has Distinct Roots: The Surface of a Polyhedron Is Flat at Every Point, Yet the Angular Defects at Its Vertices Always Sum to 4π , and to Zero on a Torus — [Paper 224]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

The surface of a polyhedron is flat at every point of it — the inside of a face is a plane, and an edge unrolls flat like paper. And the total curvature is 4π all the same. This paper shows in numbers that both of these are true. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, every regular solid gives 4π . A tetrahedron has four defects of π , a cube eight of $\pi/2$, an octahedron six of $2\pi/3$ and an icosahedron twelve of $\pi/3$ — each totalling **12.566370614359** (Section 2). Second, refining the solid does not move the total. Taking the vertices from **12** to **42** to **162** to **642**, the largest single defect falls from **1.047** to **0.0228** while the total agrees to the fourteenth digit (Section 3). Third, this is the core. Building a torus the same way gives exactly **0** — positive defects of **+12.3236** and negative ones of **-12.3236** cancelling completely (Section 5). Fourth, a negative defect could not be made by denting. Pushing the centre of a face inwards leaves its defect at **+2.384** — a dent is only a cone turned inside out, and only a saddle goes negative (Section 7). Fifth, a notch does produce one. Cutting a corner out of a cube gives three vertices of exactly $-\pi/2$, with **+17.2788** on the positive side and **-4.7124** on the negative — and the total is 4π still (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Descartes' theorem on the angular defect, Euler's polyhedron formula, the Gauss–Bonnet theorem and the equivalence of the first two are all standard. No theorem is proved — what is done here is to build polyhedra and add. No general theory of discrete differential geometry is offered. Neither the classification of surfaces nor a general treatment of genus appears — only the sphere and the torus occur. Self-intersecting polyhedra are not treated. Relation to earlier papers: Where an angular defect has appeared before, it has always been on the smooth side — Paper 27 separated the curvature of the ground from that of spacetime, and Paper 33 showed the Wigner angle to be the area of a spherical triangle. This paper is the discrete side of the same quantity, where the curvature gathers at a point. In Paper 212 every map had to distort something — here the distortion may be swept entirely onto the vertices and the bill is unchanged. What is added is adding the defects directly for four regular solids, setting the per-vertex value beside the total along a sequence of refinements, giving the ratio of defect to the area a vertex carries, measuring the cancellation on a torus, checking that denting does not make a defect negative, and obtaining three defects of $-\pi/2$ from a notch.

1. Notation and Position

The **angular defect** at a vertex of a polyhedron is 2π less all the face angles meeting there. On a flat sheet the angles come to exactly 2π and the defect is 0.

Position: Where an angular defect has appeared in earlier papers it has been on the smooth side — Paper 27 asked which manifold's curvature was meant, and Paper 33 showed the Wigner angle to be an area. This is the discrete side, where the curvature gathers at points.

2. The Regular Solids

Check 2.1: compute the angles from the coordinates and add them vertex by vertex.

Solid	Vertices	Defect at each	Total	Euler characteristic
Tetrahedron	4	π (3.141592653590)	12.566370614359	2
Cube	8	$\pi/2$ (1.570796326795)	12.566370614359	2
Octahedron	6	$2\pi/3$ (2.094395102393)	12.566370614359	2
Icosahedron	12	$\pi/3$ (1.047197551197)	12.566370614359	2

Observation 2.2

The defect at a vertex is different in all four cases, and every total is $4\pi = 12.566370614359$. The number of vertices and the defect at each balance exactly — the shape does not enter.

3. Refining the Solid Does Not Move the Total

Check 3.1: quarter each face of an icosahedron and project onto the unit sphere, repeatedly.

Vertices	Faces	Total defect	Largest single defect
12	20	12.56637061435917	1.04719755119660
42	80	12.56637061435917	0.30934133337934
162	320	12.56637061435916	0.08227110788505
642	1280	12.56637061435901	0.02276083999584

Observation 3.2

The defect at a vertex falls from 1.047 to 0.0228, a factor of forty-six, and the total agrees to the fourteenth digit. The curvature spreads thinner, and the bill never moves.

4. The Defect Approaches Curvature Times Area

On the unit sphere the curvature is 1 everywhere, so the defect at a vertex should match the area that vertex carries.

Check 4.1 (at 642 vertices):

Vertex	Defect	Area it carries	Ratio
Of degree 5	0.02276083999584	0.02268154637897	1.0034959529
Of degree 6	0.01889749080768	0.01885512890356	1.0022467046
Total	12.566370614359	12.506492733970	—

Observation 4.2

The ratio tends to 1, and the defect behaves like an integral of curvature — but note that the area comes to 12.5065 and falls short of 4π , the polyhedron being inscribed and a little small, while the defect is 4π exactly from the start. What approximates is the area, and not the total.

5. The Core: A Torus Gives Zero

Check 5.1: build a torus the same way, with $R = 2$, $r = 1$ and a 24×16 mesh.

Measured	Value
Vertices, edges, faces	384, 1152, 768
Euler characteristic	0
Total defect	0
Positive defects	+12.3235628615
Negative defects	−12.3235628615

Observation 5.2

The outside curves one way and the inside the other, and the two cancel exactly. 4π on a sphere and 0 on a torus — the total is 2π times the Euler characteristic, settled by the number of holes and not by the shape.

6. Cutting a Face Changes Nothing

Check 6.1: cut one face of a cube into four triangles about a new centre.

	Vertices	Edges	Faces	Total defect
The cube	8	12	6	12.566370614359
One face cut	9	16	9	12.566370614359

Observation 6.2

The new vertex has a defect of exactly 0 — it was placed on a flat face. Vertices, edges and faces all increased, and neither $V - E + F$ nor the total moved — a vertex is a mark for counting and not where the curvature lives.

7. Denting Fails, and a Notch Succeeds

Check 7.1 (the guess that failed): push the new centre vertex of a cube's face inwards.

Measured	Value
Defect at the pushed-in vertex	+2.38432329146618
Smallest defect anywhere on it	+0.97471550392835
Total defect	12.566370614359

Observation 7.2

The guess was wrong. Pushing a face in leaves that vertex a cone still, merely turned inside out — its angles fall short of 2π as before and the defect stays positive. A defect goes negative only at a saddle, where the angles exceed 2π .

Check 7.3: remove a corner cube from the cube $[0, 2]^3$.

Measured	Value
Vertices, edges, faces	14, 36, 24 (Euler characteristic 2)
Vertices with a negative defect	3 , each of $-\pi/2$
Positive defects	+17.278759594744
Negative defects	-4.712388980385
Total	12.566370614359

Observation 7.4

Three vertices count against the rest, and the total is 4π all the same. Convexity is not needed at all — what is needed is only that there be no hole.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of curvature is whether it is read as a value at a point or as a measure.

Where	Curvature
Inside a face	0
Along an edge	0 (the surface unrolls flat)
At a vertex	a point mass equal to the defect

Observation 8.2

As a value at a point it is zero almost everywhere, and as a measure its total is 4π . The vertices are a set of measure zero, so the integral of a function that vanishes almost everywhere comes to 4π — no contradiction, but one word standing for two different objects.

Observation 8.3 (the fence)

The fence around this surface is flat is whether a point is meant or a region. Outside it the sentence is not even false but empty — the same shape as this sequence is random in Paper 223 and the blackbody peaks here in Paper 222.

Observation 8.4

And what does not move is the real thing. The defect at a vertex shrinks by a factor of forty-six under refinement, the vertices go from 12 to 642, and a notch even changes the sign — and the total never moves once. The same shape as Paper 222, where a change of variable broke the values of a density and left the integrals alone.

9. The Census and the Close

What this paper measured:

1. Every regular solid gives 4π (Check 2.1): the per-vertex value differs in all four.
2. Refinement leaves the total (Check 3.1): the largest defect goes $1.047 \rightarrow 0.0228$.
3. The defect tends to curvature times area (Check 4.1): ratios 1.0035 and 1.0022.
4. A torus gives 0 (Check 5.1): $+12.3236$ against -12.3236 .
5. Cutting a face leaves the total (Check 6.1): the new vertex has defect 0.
6. Denting does not go negative (Check 7.1): it stays at $+2.384$.
7. A notch gives three defects of $-\pi/2$ (Check 7.3): and the total is 4π .

Close: Run a finger over the surface of a die. It is flat wherever you touch it — a face is a plane, and an edge is a single fold that goes back to paper. There is nowhere on it that curves. And adding up what is missing at the eight corners gives exactly 4π . A tetrahedron, an icosahedron, a sphere of six hundred and forty-two vertices, a cube with a corner cut away — the same 12.566370614359 every time. And punch one hole through it to make a torus, and now the answer is exactly 0. Curvature is not spread thinly somewhere: it is set down all at once on the vertices, a set of measure zero — which is why flat everywhere and a total of 4π are both right. One word carries two readings, a value at a point and a measure. And however the shape is altered, the bill keeps being presented — what can be chosen is where to put it, never how much.

References

1. Euler, L. (1758). *Elementa doctrinae solidorum*. *Novi Commentarii Academiae Scientiarum Petropolitanae*, 4.
2. Gauss, C. F. (1828). *Disquisitiones generales circa superficies curvas*. Göttingen: Dieterich.
3. Bonnet, O. (1848). Mémoire sur la théorie générale des surfaces. *Journal de l'École Polytechnique*, 19.
4. Descartes, R. (1860). De solidorum elementis. In: Foucher de Careil, A. (ed.), *Œuvres inédites de Descartes*. Paris: Auguste Durand.

5. Pólya, G. (1954). *Induction and Analogy in Mathematics*. Princeton: Princeton University Press.
6. Banchoff, T. F. (1970). Critical points and curvature for embedded polyhedra. *Journal of Differential Geometry*, 4.
7. Alexandrov, A. D. (2005). *Convex Polyhedra*. Berlin: Springer.
8. Yamagishi, Y. (2026). *We Measured the Curvature — of Which Manifold?* [Paper 27]. Zenodo. 10.5281/zenodo.22090333
9. Yamagishi, Y. (2026). *The Wigner Angle Is an Area* [Paper 33]. Zenodo. 10.5281/zenodo.22090872
10. Yamagishi, Y. (2026). *Every Map Must Distort Something* [Paper 212]. Zenodo. 10.5281/zenodo.22112104
11. Yamagishi, Y. (2026). *There Are Four Independences, and They Say Nothing to One Another* [Paper 220]. Zenodo. 10.5281/zenodo.22119838
12. Yamagishi, Y. (2026). *The Sun Peaks in the Green, in the Orange and in the Infrared* [Paper 222]. Zenodo. 10.5281/zenodo.22120199
13. Yamagishi, Y. (2026). *It Passes Every Test, and the Points Lie on Fifteen Planes* [Paper 223]. Zenodo. 10.5281/zenodo.22120728

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Yuuki YAMAGISHI (山岸 勇輝)

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