

One Ball Becomes Two of the Same Size

— Volume Has Distinct Roots: Impossible on a Line and in a Plane, Possible Only in Space, and What Separates Them Is Not the Number of Dimensions but Whether the Group Can Average — [Paper 225]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

A ball can be cut into finitely many pieces and, by rotating and rearranging them alone, made into two balls of the same size. This paper computes every computable part of that claim and shows in numbers what it is that permits it. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, two rotations generate a free group. For two rotations through $\arccos(1/3) = 1.230959417340775$, checking 354,292 words in exact integer arithmetic found not one of them equal to the identity (Section 3). Second, the reason is a single prime. Keeping a point as $(a, b\sqrt{2}, c)/3^n$, b is never once divisible by 3 — over 118,098 words, $b \bmod 3$ took only the values 1 and 2 (Section 4). Third, four pieces rebuild the whole thing twice. The 39365 words of length at most 9 split exactly into the identity and four sets of 9841, and moving two of them once each restores the whole — with 0 missing (Section 5). Fourth, this is the core. The share of a set displaced by one move falls to 0.0002 on a line and 0.01 in a plane, and in the free group it does not fall at all but rises towards 1 (Section 6). Fifth, the pieces therefore cannot have volumes. Were they to have any, $V = 2V$ and so $V = 0$, contradicting $4\pi/3 = 4.1887902048$ (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Hausdorff, Banach–Tarski, Banach's impossibility in one and two dimensions, and von Neumann's amenability are all standard. No theorem is proved — what is done here is to compute the computable parts. The decomposition itself is not constructed — constructing it requires the axiom of choice, and no decomposition can be written down. It is not shown that no Følner sequence exists — what is shown is that the most natural candidate fails, the non-existence being a theorem that is cited. The merits of the axiom of choice are not argued. Matter and physical balls are not treated — the subject is a set of points. Relation to earlier papers: Paper 209 was equal volume is not enough to cut and reassemble — this paper is its reverse: unequal volume is no obstacle, if the pieces have no volume. The argument modulo 3 that settled 209 works here unchanged. In Paper 224 the bill kept being presented however the shape was altered — here there is no bill to present. Paper 179 separated the senses of cannot be constructed — the decomposition here belongs to the sense of existing but not writable. What is added is checking 354,292 words in exact arithmetic, tabulating $b \bmod 3$ out to length 11, verifying the four-way split and the rebuilding by actual set operations and finding nothing missing, and setting the displaced share side by side for three groups.

1. Notation and Position

Let σ be a rotation about the z axis and τ one about the x axis, both through the angle $\arccos(1/3)$. A **reduced word** is a string of $\sigma^{\pm 1}$ and $\tau^{\pm 1}$ in which no two neighbours cancel.

Position: Paper 209 was equal volume is not enough to cut and reassemble — this paper is its reverse, the case where unequal volume is no obstacle at all. In Paper 224 the bill kept being presented — here there is no bill.

2. Two Rotations

Check 2.1:

Rotation	Determinant	Departure from orthogonality	Angle
About the z axis	1.0000000000000000	2.220×10^{-16}	1.230959417340775
About the x axis	1.0000000000000000	2.220×10^{-16}	1.230959417340775

$(\arccos(1/3) = 1.230959417340775.)$

Observation 2.2

Both are perfectly ordinary rotations. The only thing special about them is that the angle is incommensurable with π , and the whole matter hangs on the single fact that the cosine is $1/3$.

3. Three Hundred Thousand Words, and None the Identity

Check 3.1: keep each matrix entry as $(p + q\sqrt{2})/3^n$ with p and q whole numbers, and multiply **without rounding anything**.

Length	Reduced words	Any equal to 3^n times the identity?
1	4	none
4	108	none
7	2916	none
10	78732	none
11	236196	none
Total	354292	not one

Observation 3.2

The two rotations generate a free group — different reduced words are always different rotations. So far this is a finite check and not a proof, and the next section gives the reason.

4. The Reason Is a Single Prime

Keep the point $(1, 0, 0)$ as $(a, b\sqrt{2}, c)/3^n$. That point lies on the axis of τ , so words beginning with $\tau^{\pm 1}$ leave it where it is, and the words considered are therefore **those beginning with $\sigma^{\pm 1}$** .

Check 4.1:

Length	Words of that kind	Values taken by $b \bmod 3$
1	2	$\{1, 2\}$
5	162	$\{1, 2\}$
9	13122	$\{1, 2\}$
11	118098	$\{1, 2\}$

Observation 4.2

b is never once divisible by 3. So the point never returns and the word is never the identity — the whole business of turning one ball into two rests on a single prime. What settled the impossibility in Paper 209 was also an argument modulo 3 — the same tool, reaching the opposite conclusion.

5. Four Pieces, Rebuilt into Two

Check 5.1: form every reduced word of length at most 9 and sort them by their first letter.

Piece	Words
The identity alone	1
Beginning with σ	9841
Beginning with σ^{-1}	9841
Beginning with τ	9841
Beginning with τ^{-1}	9841
Total	39365 (and there are exactly 39365)

Check 5.2: take the piece beginning with σ together with the piece beginning with σ^{-1} moved once by σ .

Rebuilding	Words covered	Words of length ≤ 8 left uncovered
Using σ	19682	0
Using τ	19682	0

Observation 5.3

Two of the four pieces, moved once each, give back the whole. The other two do the same again — one becomes two. Nothing was created: the number of words was infinite to begin with, which is why it can appear to grow.

6. The Core: It Falls on a Line and in a Plane, and Not in the Free Group

That a group **can average** means that finite sets can be made as large as one likes which **barely move off themselves** when shifted.

Check 6.1: shift a set by one step and measure **the share displaced**.

Group	Size of the set	Share displaced	Tending to
The line	10	0.2000000000	
The line	1000	0.0020000000	
The line	10000	0.0002000000	0
The plane	25	0.4000000000	
The plane	3600	0.0333333333	
The plane	40000	0.0100000000	0
The free group	17	1.0588235294	
The free group	13121	1.0000762137	
The free group	118097	1.0000084676	1

Observation 6.2

On the line and in the plane, the larger the set the less it moves off itself. In the free group it moves off itself almost entirely, however large it is made — the share tends to 1. This is what it really is for the thing to be impossible in one and two dimensions and possible in three — what is at work is not the number of dimensions but whether the group of rotations contains a free group.

Observation 6.3

What is shown here is only that the most natural candidate fails. That every finite set fails is a theorem that is cited and not a computation of this paper.

7. So the Pieces Cannot Have Volumes

Observation 7.1

suppose the five pieces had volumes. Rotations do not change a volume, so the total after rebuilding equals the total before.

Quantity	Status
The volume of the ball	$4\pi/3 = 4.1887902048$
The volume of one piece	undefined
The total of the pieces	not a sum of numbers

Observation 7.2

Had they volumes, $V = 2V$ and so $V = 0$, contradicting $4\pi/3$. There is only one way out — that the pieces have no volume. The volume did not double: there is no such quantity in between.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of volume is which sets it is defined on.

Phrase	On which sets
The volume of the ball	the measurable ones
The volume of a piece	not defined
A finitely additive volume	on a line and a plane, on every set
The same thing	in space, on no such family

Observation 8.2 (the fence)

The fence around volume is conserved is that every set involved be measurable. Outside it the sentence is not even false but empty — the same shape as this sequence is random in Paper 223 and this surface is flat in Paper 224.

Observation 8.3

Paper 209 and this one reach opposite conclusions with the same tool.

Paper 209		This paper
The question	Does equal volume suffice?	Does unequal volume forbid?
The answer	it does not	it does not
The tool	an argument modulo 3	an argument modulo 3
The pieces	measurable (polyhedra)	not measurable

Observation 8.4

What separates them is whether the pieces can be measured. Allow only measurable pieces and volume is not enough (209) — allow unmeasurable ones and volume is not needed (here).

9. The Census and the Close

What this paper measured:

1. Both are ordinary rotations (Check 2.1): the angle is 1.230959417340775.
2. 354292 words, none the identity (Check 3.1): in exact arithmetic.
3. b is never divisible by 3 (Check 4.1): only $\{1, 2\}$ over 118098 words.
4. The split is exactly $9841 \times 4 + 1$ (Check 5.1).
5. Two pieces give back the whole (Check 5.2): nothing missing.
6. The displaced share (Check 6.1): 0.0002, 0.01, and 1.0000084676.
7. The pieces cannot have volumes (Observation 7.1): else $V = 2V$.

Close: A ball is cut into finitely many pieces, and rotating and rearranging them gives two balls of the same size. This is not a trick and not a mistake. Only one thing is happening: part-way through, the quantity called volume ceases to exist. The ball before cutting has a volume and the two afterwards have volumes — and the five pieces in between carry no such number. And what permits this is not the number three. On a line and in a plane, a set can be made large enough that shifting barely moves it, so averaging is possible — and no contradiction can arise. Inside the rotations of space there is a free group, and there every set, however large, moves off itself almost entirely. Averaging is impossible — and nothing stands in the way of the contradiction. And what guarantees that free group is only that a cosine is $1/3$ and that 3 is prime. Paper 209 said that equal volume was not enough — this one says that volume is not needed. What changed is not the ball but which pieces were allowed.

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