

There Are Five Divisions, and Each One Fails for a Different Reason

— The Numbers Always Divide and the Objects Do Not: What Separates Them Is Not the Symbol but What
Stops the Multiplication from Being Undone — [Paper 229]

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— *From thinking to syncing.* —

ABSTRACT

Division undoes a multiplication. And what stands in the way of the undoing is not of one kind. This paper counts five obstructions and shows in numbers what happens under each. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, one $12 \div 3 = 4$ is three different fours. $12\text{ m} \div 3$ is a length, $12\text{ m} \div 3\text{ m}$ is a bare count, and $12\text{ m} \div 3\text{ s}$ is a speed — what separates them is the units (Section 2). Second, in the integers division almost never works. Over 200000 random pairs the quotient landed back in the integers 1404 times, that is **0.7020%** (Section 3). Third, the divisor may have no inverse. In $\mathbb{Z}/12$, $6x = 1$ has no solution at all while $6x = 0$ has **6** — not merely absent but not unique either (Section 4). Fourth, when the multiplication does not commute there are two quotients. Over 19977 random matrix pairs the left and right quotients agreed 0 times (Section 5). Fifth, this is the core. In S_3 every subgroup has a whole-number index, and yet the quotient is a group only when the subgroup is normal — the numbers always divide and the objects do not (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Lagrange's theorem, normal subgroups and quotient groups, the Riemann sphere, IEEE 754, wheel algebras, left and right quotients in a non-commutative setting, and the partitive/quotitive distinction are all standard. No completeness is claimed — five obstructions appear and are not said to exhaust the matter. No correct definition of division by zero is chosen — which convention is better is not argued. This is not a textbook on algebra — neither localisation nor the general theory of quotient rings is treated. No position is taken on the teaching of arithmetic — the partitive and quotitive senses are cited as a distinction that already has a name. Relation to earlier papers: Paper 16 counted six roots of the equals sign, Paper 227 two kinds of inequality, Paper 228 eight plus signs — this is the fourth. Paper 228 closed by finding that only the failure to commute admits no coordinate — Section 5 here is that same obstruction appearing a second time, on the division side. Paper 181 showed that multiplication does not decide the second system — division undoes that multiplication and so inherits the ambiguity whole. What is added is separating three fours by their units, measuring how often integers divide, exhibiting both the absence and the non-uniqueness of an inverse in $\mathbb{Z}/12$, counting the agreements of the left and right quotients over 19977 pairs, and tabulating index against quotient group for every subgroup of S_3 .

1. Notation and Position

Division undoes a multiplication: $a \div b$ asks for the thing that, multiplied by b , gives a . What this paper asks is **what** stands in the way of that undoing.

Position: This is the fourth in the sorting of symbols, after Paper 16 (the equals sign), Paper 227 (inequality) and Paper 228 (the plus). The obstruction that Paper 228 found irreducible appears here a second time.

2. One Four, Three Different Things

Check 2.1:

What is asked	Expression	Answer	Is the numerator's unit kept?
Share 12 m among 3	$12\text{ m} \div 3$	4 m	yes
How many 3 m in 12 m	$12\text{ m} \div 3\text{ m}$	4	no – a bare count
12 m in 3 s	$12\text{ m} \div 3\text{ s}$	4 m/s	no – a new unit appears

Observation 2.2

The number is 4 in all three and the thing is different in all three. One is a length, one is a dimensionless count, one is a speed — what separates them is not the calculation but the units. Arithmetic teaching already names these the partitive and the quotitive.

3. The Answer Leaves the Set

Check 3.1: draw 200000 pairs of integers from 1 to 999 and count how often the quotient is again an integer.

Measured	Value
Pairs tried	200000
Quotient an integer	1404
Share	0.7020%

Observation 3.2

Inside the integers, division is simply undefined most of the time. The cure is not to adjust the operation but to enlarge the set — that is what passing from $\mathbb{Z}$ to $\mathbb{Q}$ is.

4. The Divisor Has No Inverse

Check 4.1: what $1 \div 0$ is depends on where one stands.

Standing inside	What $1 \div 0$ is
The real numbers	undefined : no such number exists
The Riemann sphere	one point , at infinity
IEEE 754	a signed infinity , and $0/0$ is not a number
A wheel	a defined element , with weaker laws
A limit	it depends on the path

Measured: $1/0 = \infty$, $-1/0 = -\infty$, $0/0 = \text{NaN}$; and x/x^2 is $+10^6$ at $x = 10^{-6}$ and -10^6 at $x = -10^{-6}$.

Check 4.2: dividing by 6 in $\mathbb{Z}/12$.

Question	Solutions
$6x = 1 \pmod{12}$	none at all
$6x = 0 \pmod{12}$	0, 2, 4, 6, 8, 10 (six of them)

The share of invertible residues is **1.0000** modulo **7**, **0.3636** modulo **12** and **0.2759** modulo **30**.

Observation 4.3

The inverse is not merely absent but not unique either. Six solutions to $6x = 0$ means that dividing 0 by 6 has six answers — a third kind of failure.

5. Without Commutativity There Are Two Quotients

Check 5.1: let A have rows $(1, 2)$ and $(3, 4)$, and B rows $(0, 1)$ and $(1, 1)$.

Which quotient	Value
$B^{-1}A$	rows (2, 2) and (1, 2)
AB^{-1}	rows (1, 1) and (1, 3)
Largest difference	1.000000

Over **19977** random matrix pairs, the left and right quotients agreed **0** times.

Observation 5.2

There is no single object called A divided by B . There is one from the left and one from the right, and they are different — the obstruction Paper 228 found irreducible is here a second time.

6. The Core: The Numbers Divide and the Objects Do Not

Check 6.1: every subgroup of the six-element group S_3 .

Subgroup H	$ H $	Index $ S_3 / H $	Normal?	Is S_3/H a group?
The identity alone	1	6	yes	yes
A_3 , the rotations	3	2	yes	yes
$\{e, (01)\}$	2	3	no	no
$\{e, (02)\}$	2	3	no	no
S_3 itself	6	1	yes	yes

Observation 6.2

The index $|G|/|H|$ is a whole number for every subgroup (Lagrange). And the quotient is a group only when H is normal — at index 3 the numbers divide cleanly while the cosets have no multiplication.

Observation 6.3

Here nothing is being inverted at all. Dividing S_3 by A_3 does not undo a multiplication; it collapses an object — under one word, one gets either a number or a thing.

7. The Census: Five Obstructions

Check 7.1:

What stops the undoing	Example	What one gets
Nothing (it is a field)	$12 \div 3$ in the rationals	a number
The answer leaves the set	$3 \div 2$ in the integers	nothing, until you enlarge
The divisor has no inverse	$1 \div 0$, or $1 \div 6$ in $\mathbb{Z}/12$	up to five answers
The product does not commute	$A \div B$ for matrices	two answers, left and right
Nothing is being inverted	$S_3 \div A_3$	an object, not a number

Observation 7.2

The five slashes share only the symbol. The first is ordinary division, the second is undefined, the third has several answers, the fourth has two, and the fifth is a different operation altogether.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of the slash is what stops the multiplication from being undone. It is not the symbol.

Observation 8.2 (the fence)

The fence around this divides is inside which set, and what is being undone. Outside it the sentence is not even false but empty — the same shape as these can be added in Paper 228 and this quantity is bounded in Paper 227.

Observation 8.3

In Paper 228, seven of eight were rescued by a change of coordinate. Here not one kind is — the obstructions to division do not vanish under relabelling. One must enlarge the set, or accept several answers, or distinguish left from right, or admit that it was a different operation.

Observation 8.4

The sorting of symbols now stands at four — six for the equals sign, two for inequality, eight for the plus, five for the slash. And it is not finished.

9. The Census and the Close

What this paper measured:

1. Three fours (Check 2.1): a length, a count, a speed.
2. Integers almost never divide (Check 3.1): 0.7020%.
3. Five answers for division by zero (Check 4.1): $1/0 = \infty$, $0/0 = \text{NaN}$, path-dependent.
4. No inverse, and not unique either (Check 4.2): six solutions to $6x = 0$.
5. Left and right quotients (Check 5.1): 0 agreements in 19977.
6. Numbers divide, objects do not (Check 6.1): index 3 with no quotient group.
7. Five obstructions (Check 7.1).

Close: $12 \div 3 = 4$. That line says at least three different things — four metres, or a bare four, or four metres every second. And inside the integers, only seven pairs in a thousand divide at all. Try to divide by zero and the answer splits five ways; try to divide by six in $\mathbb{Z}/12$ and there is either none or six. With matrices there is one answer from the left and another from the right, and in nearly twenty thousand random pairs they never once agreed. And the last of the five is not division at all. Dividing S_3 by a subgroup, the numbers come out whole every time — Lagrange

guarantees it. And yet the cosets may have no multiplication. The index is a clean 3, and the quotient, as an object, does not exist. One word, divide, was pointing both at getting a number and at collapsing a thing — so before asking whether something divides, one must ask what is being undone.

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