

Making the Curvature Constant Does Not Fix the Density

— What Yamabe's Theorem Can and Cannot Do to the Component Light Leaves Behind — [Paper 23]

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— *From thinking to syncing.* —

ABSTRACT

Paper 111 wrote that **light fixes only nine of the ten components of a metric**, and Paper 150 identified the remaining one as **a density**. **Can geometry fix that one?** Yamabe's theorem says that every conformal class contains a metric of constant scalar curvature. **This paper makes that theorem say what it can and cannot settle. NO NEW MATHEMATICAL THEOREM IS CLAIMED. First, the conformal Laplacian really is conformally covariant.** That the scalar curvature of the flat metric scaled by $u^{4/(n-2)}$ equals $-\frac{4(n-1)}{n-2}u^{-(n+2)/(n-2)}\Delta u$ was checked symbolically to **an identically vanishing difference**, and again separately for $n = 3, \dots, 7$ (Section 2). **Second, the exponent is the critical one, and $n = 2$ lies outside.** The exponent $\frac{n+2}{n-2}$ sits exactly where the Sobolev embedding loses compactness. At $n = 2$ the denominator vanishes and the question becomes a different one (Gauss–Bonnet) (Section 3). **Third, two of the corpus's numbers sit inside the Yamabe invariant of the sphere.** $\text{Vol}(S^n)$ is 4π at $n = 2$ and $2\pi^2$ at $n = 3$ — the latter being exactly the Ω_3 that Paper 46 identified as the only geometric part of $16\pi^2$ (Section 4). **Fourth, and this is the core.** **Imposing constant curvature does not fix the density.** On the sphere the solutions come in a **family**. The metric built from $u_\lambda = (\lambda/(\lambda^2 + r^2))^{(n-2)/2}$ has scalar curvature **24, 48, 80, 120** at $n = 3, 4, 5, 6$, depending on neither r nor λ (computed here). **Moving λ moves the metric while the curvature stays put** (Section 5). **Fifth, the fence is the sign of the Yamabe invariant.** Negative or zero and the solution is unique up to scale; positive and it need not be (the sphere being the example). **The separator is not whether the manifold is a sphere but the sign of one number** (Section 6). And **Yamabe holds two theorems:** the Gleason–Yamabe theorem of Paper 90, and the Yamabe problem here (Section 7).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. The conformal covariance of the conformal Laplacian, the Yamabe problem (Yamabe 1960, Trudinger 1968, Aubin 1976, Schoen 1984), the Aubin–Talenti family of solutions, and the dependence of uniqueness on the sign of the invariant are **all standard**. What this paper adds is only asking what geometry can do to the component Paper 150 said must be carried in from outside, exhibiting the failure to fix it as the disappearance of λ , writing that the fence is the sign of the invariant, and calling the roll on “Yamabe” as one person with two theorems.

1. Notation and position

For an n -manifold (M, g) with $n \geq 3$ write R_g for the scalar curvature and $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ for the conformal Laplacian, and write a conformal change as $\tilde{g} = u^{4/(n-2)}g$ with $u > 0$.

Position: Paper 111 wrote that light reaches nine, and Paper 150 identified the tenth as a density. **This paper reaches for that one from the side of curvature — and draws a line where the reach ends.**

2. The conformal Laplacian is conformally covariant

Verification 2.1: Writing out the scalar curvature of the flat metric scaled by $u^{4/(n-2)}$ from the standard formula for a conformal change,

$R_{\tilde{g}}$ from the formula	$-\frac{4(n-1)}{n-2}u^{-1-4/(n-2)}u''$
$-\frac{4(n-1)}{n-2}u^{-(n+2)/(n-2)}u''$	the same
difference	identically 0

and the difference was computed separately at $n = 3, 4, 5, 6, 7$, giving 0 each time. (QED)

Observation 2.2

Hence $R_{\tilde{g}} = u^{-(n+2)/(n-2)}L_g u$. The agreement of the exponents, $-1 - \frac{4}{n-2} = -\frac{n+2}{n-2}$, is what the covariance consists of.

3. The exponent is critical, and $n = 2$ lies outside

Verification 3.1

n	$\frac{n+2}{n-2}$	$2^* = \frac{2n}{n-2}$	note
2	—	—	the denominator vanishes: a different problem
3	5	6	critical
4	3	4	critical
5	7/3	10/3	critical
6	2	3	critical
8	5/3	8/3	critical

(QED)

Observation 3.2

The exponent of the nonlinearity sits exactly where the Sobolev embedding stops being compact. The difficulty is not a technical accident but a consequence of where the exponent is — and so is the special behaviour of the sphere.

4. Two of the corpus's numbers sit in the Yamabe invariant of the sphere

Verification 4.1: With $\text{Vol}(S^n) = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ and $Y(S^n) = n(n-1)\text{Vol}(S^n)^{2/n}$,

n	$\text{Vol}(S^n)$	numerically	$Y(S^n)$
2	4π	12.566371	25.132741
3	$2\pi^2$	19.739209	43.823233
4	$8\pi^2/3$	26.318945	61.562392
5	π^3	31.006277	78.996863
6	$16\pi^3/15$	33.073362	96.297283
7	$\pi^4/3$	32.469697	113.527275

(QED)

Observation 4.2

At $n = 3$ the volume is $2\pi^2$ — exactly the number Paper 46 identified when it wrote that of $16\pi^2$ only $\Omega_3 = 2\pi^2$ is geometry. What the corpus found inside a measure in quantum field theory sits here inside a Yamabe invariant.

Observation 4.3 (write the address)

But a shared number is not a shared root (Papers 100, 142). One is a measure on four-dimensional momentum space and the other the volume of a three-sphere. What they share is the single quantity “the volume of the unit three-sphere” — not an accident, and the same geometric object, but put to different uses.

5. Imposing constant curvature does not fix the density

Verification 5.1: On flat space take $u_\lambda(r) = \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{(n-2)/2}$ and the metric $u_\lambda^{4/(n-2)}\delta$. Then

n	R	depends on r ?	depends on λ ?
3	24	no	no
4	48	no	no
5	80	no	no
6	120	no	no

(QED) Each is the constant belonging to a sphere of radius $\frac{1}{2}$.

Observation 5.2 (this is the core)

Every member of the family has the same constant scalar curvature. λ has dropped out of the answer. Moving λ moves the metric — volumes and distances change — and the curvature does not move.

Observation 5.3

So the demand “make the scalar curvature constant” does not select one density. On the component light left behind (Paper 150), curvature places a constraint but does not supply a value.

Verification 5.4 (how large the family is): The conformal group of the sphere modulo isometries has

n	$\dim O(n+1, 1)$	$\dim O(n+1)$	free parameters
3	10	6	4
4	15	10	5
5	21	15	6
6	28	21	7

that is, $n+1$ parameters on the n -sphere. (QED)

6. The fence is the sign of the Yamabe invariant

Observation 6.1

sign of $Y(M, [g])$	what follows	uniqueness
negative	the solution is unique up to scale	fixed
zero	the solution is unique up to scale	fixed
positive	need not be unique (the sphere)	can fail

Observation 6.2

The separator is not whether the manifold is a sphere. It is the sign of the Yamabe invariant. Negative or zero, and the demand for constant curvature really does fix the density. Only when it is positive does it fail to.

Observation 6.3 (sent back to Paper 150)

Paper 150 wrote that a standard of volume must be carried in from outside. This paper corrects that by half: in a conformal class whose invariant is not positive, geometry carries it in. When it is positive, it must still come from outside.

7. One person, two theorems

Observation 7.1

theorem	content	address in the corpus
Gleason–Yamabe	Hilbert's fifth problem	Paper 90
the Yamabe problem	constant curvature in a conformal class	here

Observation 7.2

Two unrelated theorems by the same person — the shape Paper 125 found for Hopf. Writing “Yamabe's theorem” is not yet an address (Paper 17).

Observation 7.3 (the proof needed physics)

The remaining case of the Yamabe problem was closed by Schoen (1984) using the positive mass theorem of general relativity. A problem in geometry was settled by leaning on a theorem from physics — royal roads crossing (Paper 100).

8. Conclusion

1. **The conformal Laplacian is conformally covariant (Ver. 2.1): the difference vanishes identically.**
2. **The exponent is critical and $n = 2$ lies outside (Ver. 3.1).**
3. **4π and $2\pi^2$ sit in the volumes of spheres (Ver. 4.1); the latter is Paper 46's Ω_3 .**
4. **Constant curvature does not fix the density (Ver. 5.1): λ drops out of the answer.**
5. **The fence is the sign of the invariant (Obs. 6.1): if it is not positive, the density is fixed.**
6. **Yamabe holds two theorems (Obs. 7.1), and the proof needed the positive mass theorem (Obs. 7.3).**

Closing: Light teaches only the conformal class and leaves one number behind. Asking curvature to pick that number up, it sometimes can and sometimes cannot — and what separates the two cases is not the name of the manifold but the sign of a single number. What remains when it cannot is a parameter called λ , which vanishes from the answer.

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