

Every Doubling Costs the Product One Law

— Reals, Complex Numbers, Quaternions, Octonions, Sedenions: Ordering, Commutativity, Associativity, and

Then Division Itself — [Paper 231]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

What one may expect of a product depends on where one is multiplying. This paper measures which laws survive and shows that each doubling of the dimension costs exactly one. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, adding a number to itself several times is not a definition. There is no such reading of 3×2.5 — it is a picture that works in one case only (Section 2). Second, the cross product is not associative. The gap between $(a \times b) \times c$ and $a \times (b \times c)$ reached **10.7208** — and yet the Jacobi sum was 3.553×10^{-15} : it keeps a different law (Section 4). Third, this is the core. Doubling the dimension through 1, 2, 4, 8 costs exactly one law each time — ordering, then commutativity (gap 19.1), then associativity (gap 72.4) (Section 5). Fourth, at sixteen division itself breaks. $(e_1 + e_{10})(e_4 - e_{15}) = \mathbf{0}$ with both factors of length $\sqrt{2}$ — there are 336 such pairs, and not one among the octonions (Section 6). Fifth, the product rule for probabilities is not a property of multiplication. Looking at one coin twice gives $P(A)P(B) = 0.25$ against $P(A \cap B) = 0.5$ — the rule is a hypothesis about the world (Section 7).

Scope of this paper (scope note): **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED** — the Cayley–Dickson construction, the theorems of Frobenius and Hurwitz, the non-associativity of the cross product with the Jacobi identity, and the zero divisors of the sedenions are all standard. No completeness is claimed. Hurwitz's theorem is not proved — that the ladder stops at four is a theorem that is cited, and this paper only measures what falls at each step. This is not a textbook on algebra — split algebras and fields of other characteristics are not treated. The zero divisors are not classified — only those of the form $e_i \pm e_j$ are exhibited. Relation to earlier papers: This is the sixth in the sorting of symbols, after Paper 16 (six for the equals sign), 227 (two for inequality), 228 (eight for the plus), 229 (five for the slash) and 230 (seven for the minus) — completing the four operations. Paper 181 showed that multiplication does not decide the second system — its separator is the coproduct, a different cut from the one here (which law was given up). They do not overlap. Paper 220 counted four independences — Section 7 is its appearance on the multiplication side. What is added is measuring the failure of each law for matrices and the cross product, confirming to 10^{-15} that the cross product keeps Jacobi while losing associativity, measuring commutativity and associativity at each rung of the Cayley–Dickson ladder, finding a sedenion zero divisor by exhaustive search rather than by recollection, and confirming that no such pair exists among the octonions.

1. Notation and Position

Roughly four things are expected of a product: **commutativity, associativity, an answer of the same type, and distribution over addition**. What this paper measures is **which of them survive in each case**.

Position: This is the sixth in the sorting of symbols, and it completes the four operations. The separator of Paper 181 is the coproduct, a different cut.

2. Repeated Addition Is Not a Definition

Observation 2.1

Expression	As repeated addition	The product
3×4	$3 + 3 + 3 + 3 = 12$	12
3×2.5	no such reading	7.5
$3 \times (-2)$	no such reading	-6
3×0.5	no such reading	1.5

Observation 2.2

Taking a b times requires b to be a count. It is not a definition of multiplication but a picture that works in one case — the same situation as the two pictures of division in Paper 229.

3. The Product May Leave the Set It Began In

Observation 3.1

Product	Inputs	Output
$2\text{ m} \times 3\text{ m}$	two lengths	6 m^2 — an area, not a length
Dot product	two vectors	a number
Cross product	two vectors	a vector (in three dimensions only)
Wedge product	two 1-forms	a 2-form
Tensor product	dimensions 10 and 10	dimension 100

Observation 3.2

In four of the five the answer is not of the same type. Saying multiply does not say what comes out.

4. Which Laws Survive: Matrices and the Cross Product

Check 4.1: with random 3×3 matrices and random vectors in three dimensions.

Product	Commutativity gap	Associativity gap
Matrices	10.7622	1.776×10^{-15}
Cross product	$a \times b + b \times a = \mathbf{0}$ (anti-commutative)	10.7208

For the cross product the Jacobi sum measured 3.553×10^{-15} .

Observation 4.2

The cross product is not associative. And it is not lawless — in place of associativity it keeps the Jacobi identity. A law did not break; it was exchanged for another.

5. The Core: Each Doubling Costs One Law

Check 5.1: the Cayley–Dickson construction at dimensions **1, 2, 4, 8, 16**, measured over **4000** random triples at each rung.

Algebra	Dimension	Commutative	Associative
Real numbers	1	yes	yes
Complex	2	yes	yes
Quaternions	4	no (gap 19.1)	yes
Octonions	8	no (gap 24.3)	no (gap 72.4)
Sedenions	16	no (gap 36.7)	no (gap 156)

The construction was checked: for the quaternions $ij = k$, $ji = -k$ and $i^2 = -1$; for the octonions $(e_1e_2)e_4$ and $e_1(e_2e_4)$ came out with opposite signs.

Observation 5.2

Each step costs exactly one law.

Algebra	Dimension	What the doubling paid
Real numbers	1	—
Complex	2	the ordering
Quaternions	4	commutativity
Octonions	8	associativity
Sedenions	16	having no zero divisors (Section 6)

And it stops at 1, 2, 4, 8. That is Hurwitz's theorem, cited and not proved here.

6. At Sixteen, Division Itself Breaks

Check 6.1 (from recollection to exhaustive search): A pair recalled from the literature was tried first and gave $|xy| = 2.828$, not zero. A different ordering of the basis makes such a guess miss. All pairs of the form $e_i \pm e_j$ were therefore searched exhaustively.

Measured	Value
Pairs whose product is exactly zero	336
The first of them	$(e_1 + e_{10})(e_4 - e_{15})$
$ x , y $	1.4142135624, 1.4142135624
$ xy $	0
$ yx $	0
The same among the octonions	0 pairs

Observation 6.2

Neither factor is zero and the product is. Dividing by x can no longer mean anything — the third of the obstructions counted in Paper 229, the divisor having no inverse, arising here from the multiplication side. And the failure appears at sixteen and not before.

7. The Product Rule for Probabilities Is Not a Property of Multiplication

Check 7.1:

Case	$P(A)P(B)$	$P(A \cap B)$	Equal?
Two independent coins	0.25	0.25	yes
One coin looked at twice	0.25	0.5	no
An event and its complement	0.25	0	no

Observation 7.2

The product rule is not a property of multiplication. It is a hypothesis about the world, and usually a false one — the appearance on the multiplication side of the four independences of Paper 220.

8. The Separator and the Fence

Observation 8.1 (the separator)

What separates the senses of the product is which law was given up.

What was given up	Example
Nothing	the product of numbers
The ordering	the complex numbers
Commutativity	quaternions, matrices
Associativity	octonions, the cross product (Jacobi instead)
An answer of the same type	area, dot, wedge, tensor
Having no zero divisors	the sedenions
Unconditional distribution	probabilities (independence required)

Observation 8.2 (the fence)

The fence around these can be multiplied is having settled which laws are being relied on. Outside it the sentence is not even false but empty — the same shape as Papers 228, 229 and 230.

Observation 8.3

The four operations are now complete, and each has a different separator — for the plus, whether a coordinate restores it (228); for the minus, what is supposed to come back (230); for the slash, what stops the undoing (229); and for the product, which law was given up. In all four, the symbol does not say.

9. The Census and the Close

What this paper measured:

1. Repeated addition is not a definition (Observation 2.1): **no reading of 3×2.5 .**
2. Four of five change the type (Observation 3.1).
3. The cross product loses associativity and keeps Jacobi (Check 4.1): **10.7208 against 3.553×10^{-15} .**
4. Each doubling costs one law (Check 5.1): **ordering, commutativity, associativity.**
5. The recalled zero divisor was wrong (Check 6.1): **336 found by search.**
6. Division breaks at sixteen only (Check 6.1): **0 pairs among the octonions.**

7. The product rule is a hypothesis (Check 7.1): 0.25 against 0.5.

Close: Multiplication is usually taught as adding a number to itself several times. There is no such reading of 3×2.5 . And 2 metres times 3 metres is not 6 but 6 square metres, a different kind of thing. With matrices the order changes the answer, and with the cross product even the brackets do — though the cross product keeps the Jacobi identity to fifteen digits. And as the number system is doubled, exactly one law falls at each step. The complex numbers lose the ordering, the quaternions commutativity, the octonions associativity — and among the sedenions two things, neither of them zero, multiply to zero. $(e_1 + e_{10})(e_4 - e_{15}) = 0$, both factors of length $\sqrt{2}$, and 336 such pairs exist. Among the octonions there is not one. So under the single word multiply we are negotiating, place by place, what we may rely on. Before asking whether things can be multiplied, one must ask which law is being relied on — and every time one more thing is gained, one thing is let go.

References

1. Hamilton, W. R. (1844). On quaternions. *Proceedings of the Royal Irish Academy*, 3, 1–16.
2. Cayley, A. (1845). On Jacobi's elliptic functions, in reply to the Rev. B. Bronwin; and on quaternions. *Philosophical Magazine*, 26, 208–211.
3. Frobenius, G. (1878). Über lineare Substitutionen und bilineare Formen. *Journal für die reine und angewandte Mathematik*, 84, 1–63.
4. Hurwitz, A. (1898). Über die Composition der quadratischen Formen von beliebig vielen Variabeln. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, 309–316.
5. Bott, R., Milnor, J. (1958). On the parallelizability of the spheres. *Bulletin of the American Mathematical Society*, 64(3), 87–89.
6. Baez, J. C. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2), 145–205.
7. Grassmann, H. (1844). *Die lineale Ausdehnungslehre*. Leipzig: Wigand.
8. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16]. Zenodo. 10.5281/zenodo.21600471
9. Yamagishi, Y. (2026). *The Multiplication Does Not Decide the Second System* [Paper 181]. Zenodo. 10.5281/zenodo.22104155
10. Yamagishi, Y. (2026). *There Are Four Independences, and They Say Nothing to One Another* [Paper 220]. Zenodo. 10.5281/zenodo.22119838
11. Yamagishi, Y. (2026). *There Are Eight Plus Signs, and Seven of Them Become Real Addition in the Right Coordinate* [Paper 228]. Zenodo. 10.5281/zenodo.22126254
12. Yamagishi, Y. (2026). *There Are Five Divisions, and Each One Fails for a Different Reason* [Paper 229]. Zenodo. 10.5281/zenodo.22126445
13. Yamagishi, Y. (2026). *There Are Seven Minus Signs, and One of Them Undoes No Addition* [Paper 230]. Zenodo. 10.5281/zenodo.22126779

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Yuuki YAMAGISHI (山岸 勇輝)

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