

“Differentiable” Is Six Things, and Two of Them Take No Limit

— A Derivative in Every Direction with No Total Derivative, and a Fractional Derivative That Does Not Kill a Constant: the Separator Is What the Derivative Is Taken Against — [Paper 233]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

Paper 178 wrote that "both are a question of in which sense one says derivative", and stopped there. This paper is that census. “Differentiable” is at least six different claims, and two of them take no limit of difference quotients at all. NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED. First, it exists in every direction and not at all. $f(x, y) = x^3y/(x^6 + y^2)$ has directional derivative 0 in every direction, while along the curve $y = x^3$ its value stays at **0.5** however close to the origin one comes (Section 2). Second, it exists nowhere. The difference quotients of the Weierstrass function run $19.77 \rightarrow -172.32 \rightarrow -947.98 \rightarrow -7510.35 \rightarrow +11503.54$ as h shrinks, changing sign (Section 3). Third, it exists almost everywhere. The Cantor staircase has slope below 10^{-6} at all 3000 random points tested, and still climbs exactly 1 (Section 4). Fourth, some derivatives take no limit. The weak derivative of $|x|$ is **sgn**, and paired against a test function both sides gave **0.177772705925436369** (difference 6.46×10^{-27}) (Section 5). Fifth, and this is the core. The distributional derivative of the Cantor staircase is a measure and not a function — the support shrinks to length 0.0077 while the integral sits at **0.2373579604737711** (Section 6). Sixth, one derivative does not kill a constant. The Riemann-Liouville derivative of order $1/2$ returns **0.564189583548** at $x = 1$ for the constant 1 — Caputo returns 0, and the gap is exactly the initial value (Section 7).

Scope of this paper (scope note): **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED** — the Weierstrass function, the Cantor staircase, Lebesgue's differentiation theorem, Rademacher's theorem, Sobolev's weak derivative, Schwartz distributions, and the Riemann-Liouville and Caputo fractional derivatives are all standard. No ruling is made on which definition is correct — the six have different purposes, not different ranks. No application of fractional calculus is discussed; viscoelasticity and anomalous diffusion are not treated. No history of the derivative is written; who defined what and when is left to the references. No error analysis of numerical differentiation is built — the wild behaviour in Section 3 is a property of the function and not rounding, but that separation is not demonstrated here. Relation to earlier papers: Paper 178 treated the distributional derivative of a step function and stopped at the sentence naming this very question; this paper walks through that door. Paper 177 counted five distinct roots of "continuous" and used the Cantor staircase as the witness against absolute continuity; this paper looks at the same function from the side of the derivative, so the facts overlap and the question does not. Paper 197 counted four distinct roots of "converges"; this is its derivative counterpart. Paper 16 counted six distinct roots of the equals sign; the six here are roots of an operator. What is added is the measurement of the gap between directional and total differentiability, the verification of a weak derivative by pairing, the stage-by-stage convergence to the Cantor measure, the identification of the 2.59×10^{-4} discrepancy that arose in doing so, and the confirmation that the gap between the two fractional derivatives is exactly the initial value.

1. Notation and Position

After giving a step function a distributional derivative, Paper 178 wrote: **"both are a question of in which sense one says derivative."** The sentence ends there. This paper is **the census of that question.**

What is asked is **which definition reaches which object**, not which one is the real derivative.

Position: Paper 177 counted the distinct roots of continuity; this paper counts the distinct roots of the derivative. Both call the Cantor staircase as witness, and they ask it different questions — there, whether it is absolutely continuous; here, what its derivative is.

2. A Derivative in Every Direction and None in Total

Computation 2.1: put $f(x, y) = x^3y/(x^6 + y^2)$ with $f(0, 0) = 0$, and measure the directional derivatives at the origin.

direction (a, b)	$f(ta, tb)/t$ at $t = 10^{-6}$
$(0.70711, 0.70711)$	5.0×10^{-7}
$(0.70711, -0.70711)$	-5.0×10^{-7}
$(0, 1)$	0
$(1, 0)$	0
$(0.89443, 0.44721)$	1.6×10^{-6}
$(0.31623, 0.94868)$	3.33×10^{-8}

Every direction goes to 0 as $t \rightarrow 0$, so the Gâteaux derivative exists in every direction and is 0 in all of them. (QED)

Computation 2.2: now approach along a curve rather than a line.

x	$f(x, x^3)$
10^{-1}	0.5
10^{-3}	0.5
10^{-5}	0.5
10^{-8}	0.5

The value stays at 0.5 however close one comes. (QED)

Observation 2.3

f is not even continuous at the origin. Nothing discontinuous can have a total derivative, so no Fréchet derivative exists — and yet every directional derivative does. The separator is which approaches are allowed: a definition that looks only along lines throws away what the curves know.

3. A Derivative Nowhere

Computation 3.1: measure the difference quotients of $W(x) = \sum_{k \geq 0} 2^{-k} \cos(13^k \pi x)$ at $x = 0.3$.

h	$(W(x+h) - W(x))/h$
10^{-2}	19.7683562423
10^{-3}	-172.320151069
10^{-4}	-947.978382329
10^{-5}	-7510.34927975
10^{-6}	11503.5449173
10^{-7}	12914.8221772

Observation 3.2

not only does the magnitude grow, the sign flips. W is continuous everywhere and differentiable nowhere — and this does not mean that none of the six applies. The weak and distributional derivatives do exist (a continuous function is locally integrable) — what fails to exist is only the limit of the difference quotients.

4. A Derivative Almost Everywhere

Computation 4.1: build the Cantor staircase c from the self-similar definition $c(x) = c(3x)/2$ on $x \leq 1/3$, $c(x) = 1/2$ on $1/3 < x < 2/3$, $c(x) = 1/2 + c(3x - 2)/2$ on $x \geq 2/3$, and measure difference quotients with $h = 10^{-15}$ at 3000 random points.

quantity	value
$c(0)$	0
$c(1)$	1
total rise	1.0
points with slope below 10^{-6}	3000/3000 (100.00%)

Observation 4.2

a monotone function is differentiable almost everywhere (Lebesgue) — the Cantor staircase is the extreme case, with derivative 0 almost everywhere. Integrating the pointwise derivative gives 0, which does not match a rise of 1 — Paper 177 stated this as the failure of absolute continuity.

Observation 4.3

what this paper adds begins there — it is not that the derivative is absent, but that a derivative in another sense is present. That is Section 6.

5. Derivatives That Take No Limit

Definition 5.1

g is a weak derivative of f when $-\int f\varphi' = \int g\varphi$ holds for every test function φ . No difference quotient appears.

Computation 5.2: take $f(x) = |x|$ and a smooth bump φ supported on $(-0.55, 1.05)$, which contains the origin without being centred on it.

quantity	value
$-\int x \varphi' dx$	0.177772705925436369
$\int \operatorname{sgn}(x) \varphi dx$	0.177772705925436369
difference	6.46×10^{-27}
$\int 1 \cdot \varphi dx$ (a wrong candidate)	0.35519505293446355

Observation 5.3

sgn is the weak derivative of $|x|$. With the bump centred at the origin both sides vanish and nothing is decided — which is why it was moved. That the wrong candidate returns 0.355 is the evidence that this test discriminates at all.

Observation 5.4

$|x|$ has no limit of difference quotients at 0 and does have a weak derivative. The question was moved from the value at a point to what it does under an integral — the same move Paper 178 called going from value to action.

6. The Core: a Derivative That Is Not a Function

Computation 6.1: measure the distributional derivative of the Cantor staircase. By definition $\langle c', \varphi \rangle = - \int c \varphi'$. Compute the right side numerically and compare with $\int \varphi$ against the **stage- n Cantor measure**.

stage n	intervals	total length	$\int \varphi d\mu_n$
1	2	0.666666666667	0.2369516407540564
2	4	0.444444444444	0.2381665572300388
4	16	0.197530864198	0.2373636538380312
8	256	0.0390184423106	0.2373579613338792
12	4096	0.00770734662926	0.2373579604737711
18	262144	2.58×10^{-4}	0.23735796047364

Observation 6.2

the length of the support goes to 0 and the integral stays at 0.2373579604737711. The mass sits on a set of measure zero — so no density exists and this derivative is nowhere a function. Saying in Section 4 that the derivative is 0 was true only in the sense of values at points.

Computation 6.3 (a discrepancy that arose here): integrating the left side over $[0, 1]$ alone first gave **0.2370988**, disagreeing in the fourth digit.

computation	value
integrated over $[0, 1]$ only	0.2370988
the measure side	0.23735796047364
difference	2.5916×10^{-4}
$\varphi(1)$	2.5916×10^{-4}
integrated over the whole support, c extended by 0 and 1	0.2373579608640875

Observation 6.4

the discrepancy was exactly $\varphi(1)$. The test function's support runs out to 1.05 while c had been cut off at 1 — that is, no extension of c beyond its domain had been declared. The notation $\langle c', \varphi \rangle$ says nothing about this — and that is the fence of the distributional derivative: either the support lies inside the domain, or the extension is stated.

7. A Derivative That Does Not Kill a Constant

Computation 7.1: apply the derivative of order $1/2$ to the constant function $f \equiv 1$.

x	Riemann-Liouville	Caputo
0.01	5.64189583548	0
0.1	1.78412411615	0
0.5	0.797884560803	0
1	0.564189583548	0
2	0.398942280401	0

Observation 7.2

under Riemann-Liouville the derivative of a constant is not 0, and it diverges as $x \rightarrow 0$. Under Caputo it is 0 — the same phrase, "the derivative of order 1/2", returns opposite answers.

Computation 7.3: check both at $x = 1$ for $f(x) = 1 + x$ by quadrature.

definition	quadrature	closed form
Riemann-Liouville	1.69256875064	1.69256875064
Caputo	1.1283791671	1.1283791671
difference	0.564189583548	$f(0) x^{-1/2} / \Gamma(1/2)$

Observation 7.4

the gap matches $f(0) x^{-1/2} / \Gamma(1/2)$ exactly. The separator is how the initial value is handled — Riemann-Liouville integrates and then differentiates, Caputo differentiates and then integrates. Swapping the order alone decides whether a constant survives.

8. Separator and Fence

Observation 8.1 (the census)

kind	what the derivative is taken against	where it fails
Fréchet	a linear approximation uniform in all directions	the f of Section 2
Gâteaux	the limit along each direction	broken by curved approach
almost everywhere	a null set may be discarded	the Cantor staircase
weak	a pairing with test functions	$ x $ passes
distributional	the same, the answer need not be a function	c' is a measure
fractional	a convolution with a kernel	constants survive

Observation 8.2 (the separator)

what divides the six is what the derivative is taken against — values at points, directions, measures, test functions, or an integral kernel. The symbol f' names none of them.

Observation 8.3 (the fence)

there are three. First, the mode of approach: lines only, or curves as well (Section 2). Second, what may be discarded: throw away a null set and the fundamental theorem breaks (Section 4). Third, the support of the test function: if it leaves the domain, the extension must be declared (Section 6). None of the three is written in the notation.

Observation 8.4

two of the six, the weak and the distributional derivative, take no difference quotient at all. They are not definitions by a limit but requirements about integration by parts — which is why "the derivative of a non-differentiable function" is a consistent phrase. In the language of Paper 16, one symbol is serving as both a definition and a requirement.

9. Census and Closing

What this paper measured.

1. Directional derivative 0 in every direction, and still discontinuous (2.1, 2.2): 0.5 along the curve.
2. Difference quotients running wild (3.1): from 19.77 to 12914.82, sign included.
3. Slope 0 at 3000/3000 points, total rise 1 (4.1).
4. A weak derivative existing with no limit taken (5.2): both sides 0.177772705925436369.
5. A distributional derivative that is not a function (6.1): support of length 2.58×10^{-4} , integral 0.2373579604737.
6. The 2.5916×10^{-4} discrepancy was $\varphi(1)$ (6.3).
7. A fractional derivative that does not kill a constant (7.1): 0.564189583548 against 0.

Closing: Paper 178 gave a step function a derivative and then stopped at the sentence "both are a question of in which sense one says derivative." This paper counted what lies past it. "Differentiable" is six different claims, and two of them take no difference quotient at all. A function can have a derivative in every direction and none in total; it can be continuous everywhere and differentiable nowhere; it can have slope 0 almost everywhere and climb exactly 1, and its derivative can fail to be a function. And the derivative of order $1/2$ comes in two versions that answer oppositely about a constant. What divides the six is what the derivative is taken against, and the symbol f' writes none of it. The sharpest lesson here came when my own computation disagreed in the fourth digit — the discrepancy was not rounding but a test function whose support left the domain. The notation $\langle c', \varphi \rangle$ does not say how far the support runs. The fence is always outside the notation.

References

1. Weierstrass, K. (1895). Über continuirliche Functionen eines reellen Arguments. In: *Mathematische Werke*, Bd. 2. Berlin: Mayer und Müller.
2. du Bois-Reymond, P. (1875). Versuch einer Classification der willkürlichen Functionen reeller Argumente. *Journal für die reine und angewandte Mathematik*, 79.
3. Lebesgue, H. (1904). *Leçons sur l'intégration et la recherche des fonctions primitives*. Paris: Gauthier-Villars.
4. Rademacher, H. (1919). Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen. *Mathematische Annalen*, 79.
5. Gâteaux, R. (1919). Fonctions d'une infinité de variables indépendantes. *Bulletin de la Société Mathématique de France*, 47.
6. Sobolev, S. L. (1938). Sur un théorème d'analyse fonctionnelle. *Matematicheskii Sbornik*.
7. Schwartz, L. (1950--1951). *Théorie des distributions*. Paris: Hermann.
8. Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent, II. *Geophysical Journal of the Royal Astronomical Society*, 13.
9. Podlubny, I. (1999). *Fractional Differential Equations*. San Diego: Academic Press.
10. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16]. Zenodo. 10.5281/zenodo.21600471

11. Yamagishi, Y. (2026). *The Order of the Quantifiers Is the Whole Content* [Paper 177]. Zenodo. 10.5281/zenodo.22098103
12. Yamagishi, Y. (2026). *The Delta Is Not a Function but a Map* [Paper 178]. Zenodo. 10.5281/zenodo.22098261
13. Yamagishi, Y. (2026). *“Converges” Is Four Separate Claims, and What Separates Them Is What You May Discard* [Paper 197]. Zenodo. 10.5281/zenodo.22107590
14. Yamagishi, Y. (2026). *Learn from a Book with the Proofs Removed and the Equals Sign Loses Its Type* [Paper 232]. Zenodo. 10.5281/zenodo.22127528

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