

“Distance” Is Six Things, and One of Them Satisfies the Triangle Inequality Backwards

— Pseudometric, Quasimetric, Semimetric, Ultrametric, and the Lorentzian Interval: the Straight Path Becomes the Longest and Every Triangle Becomes Isosceles — [Paper 234]

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— *From thinking to syncing.* —

ABSTRACT

A metric has four axioms — non-negativity, identity of indiscernibles, symmetry, and the triangle inequality. This paper drops each of them in turn and checks in numbers what is left, and then turns the last one around without dropping it. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, identity goes. Putting $d = |x_1 - x_2|$ on the plane leaves three distinct points at distance 0 from one another (Section 2). Second, symmetry goes. On a one-way ring $d(0, 1) = 1$ while $d(1, 0) = 3$, and $d(0, 3) = 3$ while $d(3, 0) = 1$ (Section 3). Third, the triangle inequality goes. With $d = (x - y)^2$ the direct distance $d(0, 2t)$ is exactly twice $d(0, t) + d(t, 2t)$, which at $t = 5$ reads 100 against 50 (Section 4). Fourth, the triangle inequality becomes stronger. In the 2-adic distance all 19986 random triples satisfy $d(a, c) \leq \max$, and every triangle is isosceles (100.00%) (Section 5). Fifth, and this is the core. In the Lorentzian interval the triangle inequality points the other way — every detour is shorter, running $10.0 \rightarrow 9.798 \rightarrow 9.165 \rightarrow 8.0 \rightarrow 6.0$. Straight is the longest (Section 6). Sixth, satisfying all four axioms still does not fix the distance. The chordal and geodesic distances on the sphere are both genuine metrics, and their ratio runs from 1.000004 to 1.570000 (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the definitions of pseudometric, quasimetric, semimetric and ultrametric, the p -adic distance, Ostrowski's theorem, and the reversed triangle inequality of Minkowski geometry are all standard. The twin paradox is not "resolved" here; all that is claimed is that it is a statement about a metric axiom, and neither acceleration nor general relativity is treated. **No topology is discussed**; which distance generates which topology is left aside. **No taxonomy of generalised distances is built**; the six are a census and not an exhaustive list. **No p -adic analysis is entered**; the 2 -adic distance appears only as a witness for ultrametricity. **No ruling is made on which is the real distance**; Section 7 shows precisely that the axioms do not settle it. **Relation to earlier papers: Paper 201 counted four distinct roots of "complete" and showed that what ties them together is positive definiteness**; this paper counts the step before that, the distance itself. Paper 217 showed that demanding only distances be preserved drags linearity along with them; this paper counts first how many things the distance to be preserved can be. Paper 121 treated a metric on a space of metrics; this paper treats the axioms themselves. Paper 233 counted six distinct roots of "differentiable"; this is the same shape for distance. What is added is the side-by-side measurement of each axiom's failure, the isosceles rate in the 2 -adic distance, the reversed triangle inequality tabulated as a function of the detour, and the demonstration that the chordal and geodesic distances diverge by a factor of $\pi/2$.

1. Notation and Position

Definition 1.1

a function $d: X \times X \rightarrow \mathbb{R}$ is a **metric** when it satisfies four conditions.

1. **Non-negativity**: $d(x, y) \geq 0$.
2. **Identity of indiscernibles**: $d(x, y) = 0$ implies $x = y$.
3. **Symmetry**: $d(x, y) = d(y, x)$.
4. **The triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

This paper drops the second, the third and the fourth in turn, and finally changes the direction of the fourth without dropping it.

Position: Paper 201 counted the distinct roots of "complete" and showed that positive definiteness is what ties the four together — this paper is one step before that. What is asked is not which is the real distance but what stops working when each axiom is let go.

2. Pseudometric: Identity Goes

Computation 2.1: put $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$ on the plane.

p	q	$d(p, q)$
(1, 0)	(1, 5)	0
(1, 0)	(1, -3)	0
(1, 5)	(1, -3)	0
(1, 0)	(4, 0)	3
(1, 5)	(4, 0)	3

Observation 2.2

three distinct points sit at distance 0 from one another. The repair is to take a quotient, collapsing the space onto the x -axis — and the points after collapsing are not the original points. Identifying functions equal almost everywhere in L^2 is the same operation, and the same fence as discarding a null set in Section 4 of Paper 233.

3. Quasimetric: Symmetry Goes

Computation 3.1: take the one-way ring $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$ with unit weights, add an expensive shortcut $0 \rightarrow 3$ of weight 7, and measure shortest paths.

(a, b)	$d(a, b)$	$d(b, a)$	gap
(0, 1)	1	3	2
(0, 2)	2	2	0
(0, 3)	3	1	2
(1, 3)	2	2	0

Observation 3.2

getting from 0 to 3 costs 3, and getting back costs 1. The triangle inequality holds in both directions — only symmetry has gone. The repair is to symmetrise, by taking a maximum or an average, and the answers change: $d(0, 3)$ becomes neither 3 nor 1.

4. Semimetric: the Triangle Inequality Goes

Computation 4.1: put $d(x, y) = (x - y)^2$ on the line. It is non-negative, symmetric, and vanishes exactly when $x = y$.

t	$d(0, t) + d(t, 2t)$	$d(0, 2t)$	verdict
0.5	0.5	1.0	violated
1	2.0	4.0	violated
2	8.0	16.0	violated
5	50.0	100.0	violated

Observation 4.2

the detour always costs half. There is no repair here of the kind a quotient or a symmetrisation provides — taking a square root gives a metric, but that is a different function and not the same distance rescaled. Three axioms can hold and the phrase "shortest path" still lose its meaning.

5. Ultrametric: the Triangle Inequality Becomes Stronger

Computation 5.1: put the 2-adic distance $|a - b|_2 = 2^{-v_2(a-b)}$ on the integers and draw **19986** random triples of distinct integers below **4096**.

property	triples satisfying it
the ordinary triangle inequality	19986/19986 (100.00%)
the strong form $d(a, c) \leq \max(d(a, b), d(b, c))$	19986/19986 (100.00%)
the two largest sides equal (isosceles)	19986/19986 (100.00%)

Observation 5.2

nothing has been dropped here; an axiom has been strengthened. And the geometry stops looking like geometry — every triangle is isosceles, every point of a ball is its centre, and two balls that meet are nested. Strengthening an axiom is also moving to a different object.

6. The Core: the Triangle Inequality Points the Other Way

Computation 6.1: in 1+1 dimensional Minkowski spacetime join the timelike separated points $A = (0, 0)$ and $C = (10, 0)$, place $B = (5, x)$ between them, and measure $|AB| + |BC|$, where $|PQ| = \sqrt{\Delta t^2 - \Delta x^2}$.

B	$ AB $	$ BC $	$ AB + BC $
(5, 0)	5.0	5.0	10.0 (equals $ AC $)
(5, 1)	4.898979486	4.898979486	9.797958971
(5, 2)	4.582575695	4.582575695	9.16515139
(5, 3)	4.0	4.0	8.0
(5, 4)	3.0	3.0	6.0
(5, 4.9)	0.9949874371	0.9949874371	1.989974874

The straight path gives $|AC| = 10.0$. (QED)

Observation 6.2

every detour is shorter. The triangle inequality is not violated; its sign is reversed — $|AC| \geq |AB| + |BC|$. The phrase "shortest path" names the longest one.

Observation 6.3

this is the twin paradox. The twin who stays ages by 10 and the twin who turns around ages by 6 — there is no paradox anywhere, only a triangle inequality pointing the other way. That is all this paper claims; neither acceleration nor general relativity is treated.

Observation 6.4

the triangle inequality is not the only casualty — non-negativity and identity of indiscernibles fail as well (distinct points at lightlike separation have interval 0). So the Lorentzian interval is not a generalised distance but a different object — this boundary is what Paper 201 meant by saying that positive definiteness ties the four together.

7. All Four Axioms, and Still Not One Distance

Computation 7.1: on the unit sphere compare the **chord** through the surrounding space with the **great circle** along the surface. Both satisfy all four axioms.

central angle	chordal	geodesic	ratio
0.01	0.00999995833339	0.01	1.000004167
0.1	0.0999583385414	0.1	1.000416788
1	0.958851077208	1	1.042914821
2	1.68294196962	2	1.188395106
3	1.99498997321	3	1.503766956
3.14	1.99999936586	3.14	1.570000498

Observation 7.2

the ratio opens to $\pi/2$. Both are genuine metrics and the axioms choose neither — "which one is the distance" is not a theorem but a declaration of what one wants to measure. An aircraft uses the geodesic and a tunnel uses the chord.

8. Separator and Fence

Observation 8.1 (the census)

kind	axiom released or altered	what stops working
metric	none	—
pseudometric	identity of indiscernibles	telling points apart
quasimetric	symmetry	calling a round trip one quantity
semimetric	the triangle inequality	the phrase "shortest path"
ultrametric	strengthened	triangles having general shapes
Lorentzian interval	reversed	the shortest being shortest

Observation 8.2 (the separator)

what divides the six is how each of the four axioms is treated. The symbol $d(x,y)$ writes none of it — the same shape as f' in Paper 233.

Observation 8.3 (the fence)

some failures can be repaired and some cannot. A pseudometric is fixed by a quotient. A quasimetric is fixed by symmetrising, and the answers change. A semimetric is not fixed at all: the square root is a different function. And an ultrametric and the Lorentzian interval are not things to be fixed — they are different objects, not broken distances.

Observation 8.4

what Section 7 shows is that satisfying all four axioms still does not fix the distance. The fence is not inside the axioms — "which distance" can only be declared before measuring.

9. Census and Closing

What this paper measured.

1. Three distinct points at distance 0 from one another (2.1).
2. $d(0, 3) = 3$ and $d(3, 0) = 1$ (3.1).
3. The detour always costing half (4.1): 50 against 100 at $t = 5$.
4. The strong triangle inequality at 19986/19986, isosceles at 19986/19986 (5.1).
5. Every detour shorter (6.1): from 10.0 down to 1.99.
6. The chordal-to-geodesic ratio opening to 1.570000 (7.1).

Closing: a metric has only four axioms. Let them go one at a time and a named object appears each time — points stop being distinguishable, a round trip becomes two quantities, and the phrase "shortest path" loses its meaning. Strengthening changes the geometry too — in the 2-adic distance not one triangle fails to be isosceles. And there is one that drops nothing and only turns around. In the Lorentzian interval $|AC| \geq |AB| + |BC|$, and the further one detours the shorter the path. The twin who stays ages by 10, the twin who turns around by 6 — there is no paradox, only a triangle inequality pointing the other way. Finally, all four axioms together still do not fix the distance: the chord and the great circle on a sphere are both genuine metrics and their ratio opens to $\pi/2$ — which to use is a declaration, not a theorem. The symbol $d(x, y)$ says nothing about how the four axioms are being treated. As with f' in Paper 233, the fence is outside the notation.

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