

“Boundary Condition” Is Four Things, and Every Eigenvalue Moves Without a Letter of the Equation Changing

— Dirichlet, Neumann, Robin, Periodic: the Separator Is What Is Fixed at the Ends, and One of Them Loses

Existence Itself — [Paper 235]

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— *From thinking to syncing.* —

ABSTRACT

Paper 227 showed that the symbol does not settle whether a wall can be touched. This paper measures what stands next to it: what is written on the wall. The domain and the equation are left untouched, only the condition at the ends is varied four ways, and the eigenvalues are tabulated. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, one $-u'' = \lambda u$ returns four different answers. The lowest eigenvalue is **9.869617085** under Dirichlet, **0** under Neumann, **1.707053355** under Robin with $h = 1$, and **0** under periodic conditions (Section 2). Second, on an interval Dirichlet and Neumann differ by exactly one eigenvalue. Six levels agree, the largest gap being 2.88×10^{-10} — and this is an accident of one dimension: on the square **0** of **6** levels agree (Section 3). Third, Robin slides continuously between the two. As h runs from **0** to 10^6 the lowest eigenvalue moves **$0 \rightarrow 0.1967 \rightarrow 1.707 \rightarrow 6.905 \rightarrow 9.486 \rightarrow 9.8696$** , from Neumann to Dirichlet (Section 4). Fourth, and this is the core. Making the Robin coefficient negative makes the lowest eigenvalue negative — **-1.089** at $h = -0.5$ and **-100.017** at $h = -10$. The operator stops being positive (Section 5). Fifth, under periodic conditions every eigenvalue is double. After the zero mode, **39.478620540** and **39.478620541** differ by 7.1×10^{-10} (Section 6). Sixth, under Neumann conditions a solution can fail to exist. With $f \equiv 1$ a residual of 3.53×10^{-2} remains; with $f = \cos \pi x$ it falls to 5.75×10^{-12} — and even then the solution is fixed only up to a constant (Section 7).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the Dirichlet, Neumann, Robin and periodic conditions, the closed forms on an interval, the solvability condition of the Neumann problem, and the variational formulation are all standard. No numerical analysis is offered: finite elements are used as a tool, with no convergence rate and no error estimate. **No ruling is made on which condition is physically correct;** the four are different problems, not better and worse ones. **Nothing above one dimension is treated:** the square in Section 3 is a roll-call of closed forms only, and the relation between the shape of a domain and its spectrum is not entered. **Mixed and nonlinear boundary conditions are not treated;** the four are a census and not an exhaustive list. **Relation to earlier papers:** Paper 227 showed that the symbol does not settle whether a wall can be touched; this paper asks what is written on that wall. Paper 234 counted six distinct roots of distance and found the separator in how the axioms are treated; here the separator is what is fixed at the ends. Paper 233 counted six distinct roots of differentiable; this is the same shape for the boundary. Paper 125 showed that a proper name needs an address of its own, and the word "Robin" in Section 8 is an instance. What is added is the four spectra computed under one discretisation, the demonstration that on an interval the two discrete spectra coincide under a shift by one while on the square they do not, the measurement of where positivity breaks as the Robin coefficient goes negative, the confirmation of the periodic double degeneracy to 10^{-10} , and the Neumann solvability condition displayed as a residual.

1. Notation and Position

Consider $-u'' = \lambda u$ on the interval $[0, 1]$. The discretisation is linear finite elements with **800** elements, and **the same discretisation is used for all four conditions**, so every difference in the tables is a difference of condition.

The four conditions:

1. **Dirichlet:** $u(0) = u(1) = 0$ — the value at the ends is fixed.
2. **Neumann:** $u'(0) = u'(1) = 0$ — the slope, the flux, is fixed.
3. **Robin:** $u'(0) = h u(0)$ and $u'(1) = -h u(1)$ — value and slope are mixed.
4. **Periodic:** $u(0) = u(1)$ and $u'(0) = u'(1)$ — the ends are abolished.

Position: Paper 227 treated whether a wall can be touched; this paper treats what is written on the wall. What is asked is not which condition is right but how much of the interior one line at the ends can move.

2. One Domain, One Equation, Four Answers

Computation 2.1: the six lowest eigenvalues under each condition.

k	Dirichlet	Neumann	Robin ($h = 1$)	periodic
0	9.869617085	0	1.707053355	0
1	39.478620540	9.869617085	13.492380850	39.478620540
2	88.827466976	39.478620540	43.357465877	39.478620541
3	157.916917414	88.827466976	92.770469520	157.916917413
4	246.748037301	157.916917414	161.884268244	157.916917414
5	355.322196528	246.748037301	250.727077848	355.322196527

The closed forms are $(k + 1)^2\pi^2$, $k^2\pi^2$ and $(2m)^2\pi^2$, with $\pi^2 = 9.869604$. (QED)

Observation 2.2

the differential equation was not changed by one letter. Only one line at the ends was changed, and every eigenvalue moved — so a boundary condition is not an accessory to an equation but part of it.

3. On an Interval the Two Differ by Exactly One Eigenvalue

Computation 3.1: match the $(k+1)$ th Neumann eigenvalue against the k th Dirichlet one.

k	Neumann ($k+1$)	Dirichlet (k)	difference
0	9.869617085	9.869617085	6.29×10^{-11}
1	39.478620540	39.478620540	1.53×10^{-10}
2	88.827466976	88.827466976	2.88×10^{-10}
3	157.916917414	157.916917414	2.74×10^{-10}
5	355.322196528	355.322196528	9.69×10^{-11}

Observation 3.2

on an interval, what separates Dirichlet from Neumann is a single eigenvalue: the constant function, at zero.

Computation 3.3 (the fence): on the unit square both spectra are $\pi^2(m^2 + n^2)$, with $m, n \geq 1$ for Dirichlet and $m, n \geq 0$ for Neumann.

k	Neumann ($k+1$)	Dirichlet (k)	equal?
0	9.869604	19.739209	no
1	9.869604	49.348022	no
2	19.739209	49.348022	no
3	39.478418	78.956835	no
5	49.348022	98.696044	no

Observation 3.4

0 of 6. So the agreement of Section 3.1 is an accident of one dimension and not a theorem about the conditions. A relation seen in one dimension must not be carried upward — that is the fence of this section.

4. Robin Slides Continuously Between the Two

Computation 4.1: vary h and measure the lowest eigenvalue.

h	lowest eigenvalue	approaching
0	0	Neumann
0.1	0.196710694	
1	1.707053355	
10	6.904684389	
100	9.486484922	
10^6	9.869577601	Dirichlet ($\pi^2 = 9.869604$)

Observation 4.2

the two conditions are the ends of one line, not two separate points. But the ends are limits and not any particular h — the same shape as the touched and untouched walls of Paper 227.

5. The Core: a Negative Coefficient Costs Positivity

Computation 5.1: push h below zero.

h	lowest eigenvalue	sign
0	0	zero
−0.5	−1.089156943	negative
−1	−2.382097139	negative
−2	−5.756911044	negative
−5	−25.640243768	negative
−10	−100.016842629	negative

Observation 5.2

the operator $-u''$ did not change by one letter. Only one line at the ends changed, and positivity broke — because the boundary term in the form $\int |u'|^2 + h(u(0)^2 + u(1)^2)$ now pulls downward.

Observation 5.3

the sentence "the Laplacian is a positive operator" has a fence omitted — the correct statement names under which boundary condition. In the language of Paper 16, outside its fence that sentence is not even false; it is empty.

6. With No Ends, Everything Comes in Pairs

Computation 6.1: periodic eigenvalues and the gap to the next.

k	eigenvalue	gap to the next
0	0	3.95×10^1
1	39.478620540	7.09×10^{-10}
2	39.478620541	1.18×10^2
3	157.916917413	5.10×10^{-10}
4	157.916917414	1.97×10^2

Observation 6.2

after the zero mode the eigenvalues come in pairs. With no ends, a wave going right costs what a wave going left costs — and the 10^{-10} is the rounding of the discretisation, not a genuine splitting.

7. Under Neumann Conditions a Solution Can Fail to Exist

Computation 7.1: solve $-u'' = f$ with $u'(0) = u'(1) = 0$ in least squares and read the residual.

f	$\int_0^1 f$	residual
1	1	3.53×10^{-2}
$\cos \pi x$	0	5.75×10^{-12}

The matrix has rank **800** out of **801**, and **its kernel is the constants**. (QED)

Observation 7.2

with $f \equiv 1$ no solution exists. One cannot hold the flux at zero on both ends while allowing a net source inside — so the solvability condition $\int f = 0$ is imposed by the boundary condition and not by the equation.

Observation 7.3

and when a solution does exist, it is fixed only up to a constant. Under Dirichlet conditions existence and uniqueness are unconditional; under Neumann conditions existence acquires a condition and uniqueness is lost — the same quotient that Paper 234 called the repair for a pseudometric is needed here.

8. Separator and Fence

Observation 8.1 (the census)

condition	what is fixed at the ends	what is lost
Dirichlet	the value	nothing
Neumann	the slope, the flux	existence and uniqueness
Robin	a mixture of both	positivity (when $h < 0$)
periodic	the ends themselves	simplicity (all doubled)

Observation 8.2 (the separator)

what divides the four is what is fixed at the ends. The symbol $-u'' = \lambda u$ names none of them — the same shape as f' in Paper 233 and $d(x, y)$ in Paper 234.

Observation 8.3 (the fence)

there are three. First, positivity is not a property of the equation but of the boundary condition (Section 5). Second, neither are existence and uniqueness: both fail under Neumann conditions (Section 7). Third, a relation seen in one dimension must not be carried upward: the agreement on the interval holds at 0 of 6 levels on the square (Section 3).

Observation 8.4 (on the name)

it has been pointed out that no trace has been found of Robin himself treating this condition (Gustafson and Abe). This is the kind of case Paper 125 had in mind in writing that a proper name needs an address — this paper does not adjudicate attribution: it follows the usual name and records only that a name does not guarantee a provenance.

9. Census and Closing

What this paper measured.

1. One equation, four answers (2.1): lowest eigenvalues 9.8696, 0, 1.7071, 0.
2. On the interval, a difference of one eigenvalue (3.1): agreeing to 2.88×10^{-10} .
3. On the square, 0 of 6 levels agree (3.3).
4. Robin sliding continuously (4.1): from 0 to 9.869577601.
5. A lowest eigenvalue of -100.017 at a negative coefficient (5.1).
6. Every periodic eigenvalue double (6.1): gap 7.09×10^{-10} .
7. No Neumann solution for a source with nonzero integral (7.1): residual 3.53×10^{-2} .

Closing: the domain was not changed, the equation was not changed by one letter, and only one line at the ends was varied four ways. With that, every eigenvalue moved, under one condition the operator stopped being positive, and under another a solution stopped existing. So "the Laplacian is positive" and "the solution is unique" are not

statements about an equation — both are statements about a boundary condition, and without one written down they are not even false, but empty. Paper 227 wrote that the symbol does not settle whether a wall can be touched. What this paper adds is that the symbol does not settle what is written on the wall either. And finally, the agreement that held on an interval held at not one level on the square — carrying a relation upward in dimension needs a separate reason to be allowed.

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