

One and the Same “= 0” Returns Five Different Kinds of Collection

— $x^2 = -1$ Is Empty over the Reals, Two Points over the Complexes and a Two-Sphere over the Quaternions:
the Separator Is Where the Unknown Is Allowed to Live, and Stationary Does Not Mean Minimal — [Paper
236]

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— *From thinking to syncing.* —

ABSTRACT

When one says "solve", the usual first question is how many answers there are. This paper asks instead what kind of object the collection of answers is — an empty set, a pair of points, a sphere, a vector space, or a single function that is not even a minimum. Every equation here is written with the same " $= 0$ ". **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, one $x^2 = -1$ returns three different objects. Nothing over the reals, two points over the complexes, and a two-dimensional sphere over the quaternions — across 20000 unit purely imaginary quaternions the largest component of $q^2 + 1$ was 1.11×10^{-16} (Section 2). Second, finite fields change it too. $x^2 + 1 = 0$ has two solutions modulo 5, 13, 17, 29 and none modulo 3, 7, 11, 19, 23 (Section 3). Third, for a matrix it becomes a surface. The real 2×2 matrices with $X^2 = I$ form a two-parameter family (Section 4). Fourth, for a differential equation the collection is a vector space. $y' - y = 0$ gives dimension one and $y'' + y = 0$ gives dimension two, with Wronskian 1.000000000000 (Section 5). Fifth, and this is the core. For a variational problem " $= 0$ " means stationary and not minimal. For $J[y] = \int_0^L (y'^2 - y^2)$ the function $y \equiv 0$ is stationary for every L , and yet $J = +0.1449$ at $L = 3$, -0.3401 at $L = 3.5$ and -4.5065 at $L = 10$: at exactly $L = \pi$ a minimum turns into a saddle (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the count of roots over an algebraically closed field, the solution set of $x^2 = -1$ in the quaternions, the elementary consequence of quadratic reciprocity used here, the dimension of the solution space of a linear ordinary differential equation, and the classification of stationary points by the second variation are all standard. No existence theorem is proved; every section only counts solutions already known. **Galois theory is not entered**; Section 3 counts and does not explain. **Nonlinear differential equations are not treated**; "the solution space is a vector space" holds in the linear case only. **No general theory of the calculus of variations is built**; Section 6 examines one functional at one stationary point. No numerical analysis is offered. **Relation to earlier papers:** Paper 231 showed that each step down the Cayley-Dickson ladder costs one law; this paper looks at the same ladder from the side of the solutions of an equation. Paper 16 counted six distinct roots of the equals sign; this paper splits the inside of one of those uses, the conditional. Paper 179 counted distinct roots of "cannot be constructed" and touched Cauchy's functional equation; this paper does not go there. Paper 227 asked whether an equals sign is ever attained; Section 6 is its variational counterpart. Paper 235 measured a lowest eigenvalue of π^2 from the side of boundary conditions; the $L = \pi$ of Section 6 arrives at the same π^2 from the other side. What is added is the verification with twenty thousand samples that the quaternionic solution set is a sphere, the count of the same polynomial's roots over nine moduli, the residual check of the two-parameter family of matrix solutions, the measurement of solution-space dimension by the Wronskian, and the location of the point $L = \pi$ where a stationary point stops being a minimum.

1. Notation and Position

The question: when one writes $F(x) = 0$, what kind of object is the collection $\{x : F(x) = 0\}$?

This is not a question about how many. An empty set, a pair of points, a sphere, a vector space and a single function differ in kind before they differ in number. The question cannot be answered without deciding where the unknown is allowed to live — and the symbol " $= 0$ " does not write that down.

Position: Paper 16 counted six distinct roots of the equals sign itself; this paper splits the inside of one of them, the conditional.

2. Change the Numbers and the Kind Changes

Computation 2.1: solve $x^2 = -1$ in three places.

where the unknown lives	the solution set	kind
the reals $\mathbb{R}$	none	the empty set
the complexes $\mathbb{C}$	$\pm i$	two points
the quaternions $\mathbb{H}$	all unit purely imaginary ones	a two-sphere

Computation 2.2: draw 20000 unit purely imaginary quaternions at random and square them.

quantity	value
largest component of $q^2 - (-1)$	1.11×10^{-16}
mean length of the imaginary part	1.000000000000
rank of the point cloud in $\mathbb{R}^3$	3

All twenty thousand are solutions; their imaginary parts lie on the unit sphere, and that sphere spans $\mathbb{R}^3$. (QED)

Observation 2.3

one quadratic returned the empty set, two points, and a two-sphere. The sentence "a quadratic has at most two roots" has a fence omitted — it holds over a commutative field. Paper 231 counted the rung at which commutativity is lost; this is what it costs.

3. Finite Fields Change It Too

Computation 3.1: solve $x^2 + 1 = 0$ modulo p .

p	solutions	count
3	none	0
5	2, 3	2
7	none	0
11	none	0
13	5, 8	2
17	4, 13	2
19	none	0
29	12, 17	2

Observation 3.2

two solutions when the prime leaves remainder 1 on division by 4, none when it leaves 3. This paper counts and does not explain — the point here is only that the same polynomial's " $= 0$ " has a different collection at each address.

4. For a Matrix It Becomes a Surface

Computation 4.1: solve $X^2 = I$ among real 2×2 matrices. Every X whose rows are (a, b) and $((1 - a^2)/b, -a)$ is a solution, for any a and any $b \neq 0$.

a	b	$\ X^2 - I\ $
0	1	0
0.5	2	0
2	3	0
-1.5	0.4	1.24×10^{-16}
7	0.25	0

Together with $\pm I$ these are all of them. (QED)

Observation 4.2

the solutions are a two-parameter surface, not a list one can write out. " $X^2 = I$ implies $X = \pm I$ " is a statement about numbers and not about matrices — the fence is the absence of zero divisors.

5. For a Differential Equation the Collection Is a Vector Space

Computation 5.1: integrate to $t = 1$ with fourth-order Runge-Kutta.

equation	the solution set	dimension
$y' - y = 0$	ce^t (2.718281828 and 5.436563657 proportional)	1
$y'' + y = 0$	$c_1 \cos t + c_2 \sin t$ (Wronskian 1.000000000000)	2

The computed values matched $\cos 1 = 0.540302306$ and $\sin 1 = 0.841470985$. (QED)

Observation 5.2

here " $= 0$ " returns neither points nor a surface but a vector space. And its dimension equals the order of the equation — the fence is linearity, and in the nonlinear case this way of speaking fails (not treated here).

6. The Core: Stationary Does Not Mean Minimal

Computation 6.1: let $J[y] = \int_0^L (y'^2 - y^2)dx$ with $y(0) = y(L) = 0$. The function $y \equiv 0$ makes the first variation vanish for every L . Feed in $y = \sin(\pi x/L)$ and measure J .

L	$J[\sin(\pi x/L)]$	the type of $y \equiv 0$
1	4.434802201	minimum
2	1.467401100	minimum
3	0.144934067	minimum
π	0	degenerate
3.5	−0.340056514	saddle
4	−0.766299450	saddle
10	−4.506519780	saddle

Numerical quadrature agreed with the closed form $\frac{L}{2}((\pi/L)^2 - 1)$ at every row. (QED)

Observation 6.2

$y \equiv 0$ satisfies " $= 0$ " for every L . And for $L > \pi$ it is a saddle and not a minimum — "the variation vanishes, therefore it is a minimum" is false.

Observation 6.3

the turn happens at exactly $L = \pi$, which is where the lowest Dirichlet eigenvalue π^2/L^2 crosses 1 — Paper 235 arrived at the same π^2 from the side of boundary conditions; this paper arrives from the side of the variation.

Observation 6.4

and what a variational " $= 0$ " returns is neither a number nor a point but a single function — and whether that function gives a minimum is decided outside the " $= 0$ ". The same shape as Paper 227 writing that an equals sign does not settle whether it is attained.

7. Separator and Fence

Observation 7.1 (the census)

where the unknown lives	the solution set	kind
a commutative field	at most deg points	a finite set
a noncommutative division ring	the unit imaginary quaternions	a sphere, a manifold
a finite field	0 or 2	finite, depending on the modulus
a matrix ring	a two-parameter family	a surface (zero divisors exist)
a linear differential operator	dimension equal to the order	a vector space
a functional	the stationary function	one function, not necessarily minimal

Observation 7.2 (the separator)

what divides the six is where the unknown is allowed to live. The symbol " $= 0$ " names none of them — the same shape as f' in Paper 233, $d(x, y)$ in Paper 234 and $-u''$ in Paper 235.

Observation 7.3 (the fence)

three familiar sentences each have a fence omitted. "A degree n equation has at most n roots" — over a commutative field (Section 2). " $X^2 = I$ implies $X = \pm I$ " — where there are no zero divisors (Section 4). "The variation vanishes, so it is a minimum" — only when the second variation is positive (Section 6). Outside their fences all three are not even false, but empty.

8. Census and Closing

What this paper measured.

1. $x^2 = -1$ giving the empty set, two points and a sphere (2.1, 2.2): twenty thousand samples, error 1.11×10^{-16} .

2. The same polynomial giving 0 or 2 depending on the modulus (3.1).
3. Matrix solutions forming a two-parameter surface (4.1).
4. Solution-space dimension equal to the order (5.1): Wronskian 1.000000000000.
5. A stationary point turning into a saddle at $L = \pi$ (6.1): from +0.1449 to −0.3401.

Closing: the word "solve" sounds like a question about how many answers there are. And yet one $x^2 = -1$ returned the empty set, two points and a two-dimensional sphere — objects that differ in kind. Put the unknown in a matrix ring and the answer is a surface; in a linear differential operator and it is a vector space; in a functional and it is one function. What divides them is where the unknown is allowed to live, and the symbol " $= 0$ " does not write it. The last one bites hardest — a variational " $= 0$ " means stationary and not minimal. The function $y \equiv 0$ is stationary for every L , and for $L > \pi$ it is a saddle. The turning point $L = \pi$ is the same number Paper 235 measured from the side of boundary conditions — two roads arriving at one fence from opposite sides.

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