

“Infinitesimal” Is Five Things, and Squaring Them Gives Three Different Answers

— The Square Is Smaller and Nonzero, Exactly Zero, or a First-Order Quantity: and Two of the Five Are Not Small at All — [Paper 237]

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— *From thinking to syncing.* —

ABSTRACT

At least five things are called infinitesimal. The sharpest way to tell them apart is to square them — the square comes out smaller and nonzero, exactly zero, or a first-order quantity. And two of the five are not small at all. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, dx is not small. For $f(x) = x^3$ at $a = 2$ the gap between increment and differential, divided by h , falls from 6.00×10^{-1} to 6.00×10^{-5} , while the coefficient $3a^2 = 12$ never moves (Section 2). Second, a genuine infinitesimal exists. In the ordered field of rational functions $\varepsilon = 1/t$ is a positive element below every $1/n$, and its square is not zero but a second-order infinitesimal (Section 3). Third, and this is half the core. In the dual numbers ε^2 is exactly zero — so $f(a + \varepsilon)$ returns the derivative with error 0, while the difference quotient bottoms out at 7.29×10^{-8} for $h = 10^{-8}$ and then climbs back to 9.93×10^{-7} (Section 4). Fourth, the variational δ is not small either. It is a linear functional of the direction: at $L = 4$ the second variation is -0.766299449864 along $\sin(\pi x/L)$ and $+2.934802200545$ along $\sin(2\pi x/L)$ (Section 5). Fifth, one infinitesimal has a first-order square. For Brownian motion $\sum(\Delta W)^2$ runs $0.932 \rightarrow 0.978 \rightarrow 0.9996 \rightarrow 1.0013 \rightarrow 1.0008$, going to $T = 1$, while $\sum(\Delta t)^2$ falls from 10^{-2} to 10^{-6} (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — the linearity of the differential, the ordered field of rational functions as a non-Archimedean example, the dual numbers and forward-mode automatic differentiation, the first and second variations, and the quadratic variation of Brownian motion are all standard. **Nonstandard analysis is not constructed here:** Section 3 shows only that a non-Archimedean ordered field exists, and neither the transfer principle nor an ultrapower is treated. **Synthetic differential geometry is not entered;** Section 4 uses the dual numbers as one ring for one computation. **No theory of automatic differentiation is written;** the comparison is one function at one point. **No general theory of stochastic analysis is built;** Section 6 looks at the quadratic variation only. **The question whether infinitesimals exist is not answered;** all this paper says is **name the system and the element.** **Relation to earlier papers:** Paper 148 wrote that $\varepsilon^2 = 0$ is the definition of a ring while $(dW)^2 = dt$ is a fact about a path; this paper puts the other three between them and makes it a census. Paper 132 showed that Itô's 2 is not a dimension; Section 6 borrows only its quadratic-variation side. Paper 233 counted six distinct roots of differentiable and called the Gâteaux derivative a limit along each direction; the δ of Section 5 is exactly that. Paper 236 showed that a variational " $= 0$ " is stationary and not minimal; this paper asks what that δ itself is. What is added is the demonstration that the gap between increment and differential vanishes even after division by h , the check of non-Archimedean order against $1/n$, the side-by-side comparison of dual-number and finite-difference derivatives showing the latter degrade past 10^{-8} , the sign change of the second variation with direction, and the five-decade tracking of the quadratic variation against the vanishing of $\sum(\Delta t)^2$.

1. Notation and Position

Paper 148 wrote that $\varepsilon^2 = 0$ is the definition of a ring while $(dW)^2 = dt$ is a fact about a path. That one sentence sets two infinitesimals side by side — this paper puts the other three between them and turns it into a census.

Method: comparing sizes is useless (all five are called small). **Squaring is the sharp test — and the answer splits three ways.**

2. dx Is Not Small

Computation 2.1: with $f(x) = x^3$ and $a = 2$, compare the increment $f(a + h) - f(a)$ with the differential $3a^2h$.

h	increment	differential	gap $\div h$
10^{-1}	6.610000000000	6.000000000000	6.100000×10^{-1}
10^{-2}	1.206010000000	1.200000000000	6.010000×10^{-2}
10^{-3}	0.120060010000	0.120000000000	6.001000×10^{-3}
10^{-5}	0.001200060000	0.001200000000	6.000010×10^{-5}

Observation 2.2

the gap goes to zero even after division by h . So the differential is not an approximation to a small number but the linear function of h whose error is $o(h)$ — and the coefficient $3a^2 = 12$ never shrinks. The phrase " dx is an infinitely small number" names nothing in this system.

3. A Genuine Infinitesimal: a Non-Archimedean Ordered Field

Computation 3.1: order the field of rational functions by declaring $f > 0$ when $f(t) > 0$ for all large enough t , and set $\varepsilon = 1/t$.

n	$\varepsilon < 1/n$ from $t =$	the comparison there
1	2.0	$0.500000000 < 1.000000000$
10	11.0	$0.090909091 < 0.100000000$
100	101.0	$0.009900990 < 0.010000000$
1000	1001.0	$0.000999001 < 0.001000000$

So $0 < \varepsilon < 1/n$ holds for every n , and the field is **not Archimedean**. (QED)

Computation 3.2: look at the square.

t	ε	ε^2
10	1.0×10^{-1}	1.0×10^{-2}
10^3	1.0×10^{-3}	1.0×10^{-6}
10^6	1.0×10^{-6}	1.0×10^{-12}
10^9	1.0×10^{-9}	1.0×10^{-18}

Observation 3.3

ε^2 is not zero. It is a second-order infinitesimal, below every ε — and here it parts company with the ε of the next section.

Observation 3.4 (scope)

what this section showed is only that a non-Archimedean ordered field exists. The power of nonstandard analysis lies in the transfer principle, and that has not been shown here — what has not been shown must not be written as though it had.

4. Half the Core: an Infinitesimal Whose Square Is Exactly Zero

Computation 4.1: implement the dual numbers $a + b\varepsilon$ with $\varepsilon^2 = 0$ and multiply ε by itself.

quantity	value
ε	$0 + 1\varepsilon$
$\varepsilon \cdot \varepsilon$	$0 + 0\varepsilon$

Computation 4.2: read the ε coefficient of $f(2 + \varepsilon)$.

f	dual-number $f'(2)$	exact $f'(2)$	error
x^3	12.000000000000000	12	0
$x^3 + 5x$	17.000000000000000	17	0
$x \cdot x \cdot x \cdot x$	32.000000000000000	32	0

Computation 4.3: get the same derivative from a difference quotient.

h	$(f(2+h) - f(2))/h$	error against 12
10^{-4}	12.000600010023	6.000×10^{-4}
10^{-6}	12.000006002211	6.002×10^{-6}
10^{-8}	11.999999927070	7.293×10^{-8}
10^{-9}	12.000000992884	9.929×10^{-7}
10^{-11}	12.000000992884	9.929×10^{-7}

Observation 4.4

the error of the difference quotient bottoms out at $h = 10^{-8}$ and then rises — rounding overtakes truncation. The dual numbers choose no h at all, so the error is 0. One word, and one side takes a limit while the other does not — the same shape as the two of Paper 233 that take no difference quotient.

5. The Variational δ Is Not Small Either

Computation 5.1: with $J[y] = \int_0^L (y'^2 - y^2)$, $L = 4$ and $y \equiv 0$, measure the first and second variations along each direction v .

direction v	first variation	second variation
$\sin(\pi x/L)$	0	−0.766299449864
$\sin(2\pi x/L)$	0	+2.934802200545
$\sin(3\pi x/L)$	0	+9.103304951226

Observation 5.2

δJ is not a small number but a linear functional of the direction. It is the Gâteaux derivative of Paper 233 — so " δy is a tiny change" invites the same misreading as dx .

Observation 5.3

the second variation changes sign with the direction: at $L = 4$ it is negative along the first and positive after — which is what Paper 236 meant by a saddle for $L > \pi$.

6. The Core: an Infinitesimal Whose Square Is First Order

Computation 6.1: split Brownian motion on $[0, 1]$ into N steps and measure the quadratic variation.

N	$\sum(\Delta W)^2$	$\sum \Delta W ^3$	$\sum(\Delta t)^2$
10^2	0.932271698	1.324×10^{-1}	1.0×10^{-2}
10^3	0.977724833	5.101×10^{-2}	1.0×10^{-3}
10^4	0.999646793	1.586×10^{-2}	1.0×10^{-4}
10^5	1.001253242	5.068×10^{-3}	1.0×10^{-5}
10^6	1.000780569	1.598×10^{-3}	1.0×10^{-6}

Observation 6.2

$\sum(\Delta W)^2$ goes to $T = 1$ and does not vanish, while $\sum(\Delta t)^2$ and $\sum|\Delta W|^3$ both do — so $(\Delta W)^2$ behaves as a first-order and not a second-order quantity. This is where Itô's correction comes from.

Observation 6.3

the infinitesimal here is neither the definition of a ring nor an element of an ordered field, but a property of a path — Paper 148's contrast between convention and fact appears directly as a difference in what the square is.

7. Separator and Fence

Observation 7.1 (the census)

what it is called	its square	what it is
dx	—	a linear form (not small)
ϵ in a non-Archimedean field	nonzero, second order	an element of a field
ϵ in the dual numbers	exactly zero	the definition of a ring
the variational δ	—	a linear functional (not small)
dW	the first-order dt	a property of a path

Observation 7.2 (the separator)

what divides the five is what the square comes out to be, and the answer runs three ways — smaller and nonzero, exactly zero, or first order. And the remaining two are not things one squares at all — both dx and δ are linear maps.

Observation 7.3 (the fence)

the word "infinitesimal" always needs an address. Which system, and which element — an element of an ordered field, the defining relation of a ring, an element of a cotangent space, a functional, or a property of a path. Without the address the word is not even false, but empty (Paper 16).

8. Census and Closing

What this paper measured.

1. The gap between increment and differential vanishing after division by h (2.1): the coefficient 12 never moves.
2. $\epsilon < 1/n$ for every n (3.1), and ϵ^2 nonzero (3.2).
3. $\epsilon^2 = 0$ exactly and derivative error 0 in the dual numbers (4.1, 4.2).
4. The difference quotient bottoming out at 10^{-8} and degrading after (4.3): $7.29 \times 10^{-8} \rightarrow 9.93 \times 10^{-7}$.
5. The second variation changing sign with direction (5.1): -0.7663 and $+2.9348$.
6. $\sum(\Delta W)^2 \rightarrow 1$ while $\sum(\Delta t)^2 \rightarrow 0$ (6.1).

Closing: five things called infinitesimal were set out and squared. The answers split three ways — smaller and nonzero in a non-Archimedean field, exactly zero in the dual numbers, and the first-order dt for Brownian motion. And the other two, dx and δ , are not small at all. Both are linear maps rather than quantities — whatever "an infinitely small number" is meant to name, it does not exist in those two. Paper 148 wrote that $\epsilon^2 = 0$ is the definition of a ring while $(dW)^2 = dt$ is a fact about a path. What this paper adds is that three more sit between them, and squaring tells which is which. Without an address the word is not even false, but empty.

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