

“Regular” Is Three Things, and in the Complex Plane One Derivative Gives Infinitely Many

— On the Real Line Once Does Not Give Twice and Infinitely Often Does Not Give Analytic: the Separator Is Which Directions of Approach Are Allowed — [Paper 238]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

Paper 233 showed that a derivative in every direction does not give a total derivative. This paper measures the other side: strengthen the demand about direction by one step and infinite differentiability and analyticity arrive together, unasked. And the one word "regular" names three different things. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, on the real line once does not give twice. For $f(x) = x|x|$ the second difference quotient is $+2$ from the right and -2 from the left, and shrinking h never brings them together (Section 2). Second, on the real line infinitely differentiable does not give analytic. For $g(x) = e^{-1/x^2}$ the ratios $g(0.1)/h^{10} = 3.72 \times 10^{-34}$ and $g(0.1)/h^{20} = 3.72 \times 10^{-24}$ show it beats every power — so every derivative at 0 vanishes, the Taylor series is identically 0, and yet $g(0.5) = 1.83 \times 10^{-2}$ (Section 3). Third, and this is the core. In the complex plane the values on a circle alone give every derivative. For e^z and n from 0 to 6, Cauchy's formula differed from the exact value by between 1.24×10^{-16} and 7.85×10^{-14} (Section 4). Fourth, the maximum always sits on the boundary. On the unit disc $|e^z|$ reaches **2.711477622183** inside and **2.718281828459**, which is e , on the boundary — so on a compact surface without boundary the only holomorphic functions are constants (Section 5). Fifth, "regularity" in partial differential equations is a third word. For $-u'' = f$ a smooth source leaves neighbouring values of u'' differing by 0.001571, and a source with one jump leaves them differing by **1.000000** (Section 6).

Scope of this paper (scope note): **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED** — the Cauchy-Riemann equations, Cauchy's integral formula, the maximum modulus principle, Liouville's theorem, the non-analyticity of e^{-1/x^2} , and elliptic regularity are all standard. **Complex analysis is not rebuilt**: Section 4 checks a formula numerically and proves nothing. **Algebraic geometry is not entered**: Section 5 shows only the maximum modulus principle, with no sheaves and no schemes. **No elliptic regularity theorem is proved**; Section 6 counts one example in one dimension. **No ruling is made on which regularity is better**; the three have different purposes. **No numerical analysis is offered**. **Relation to earlier papers**: Paper 233 showed that a derivative in every direction does not give a total derivative; this paper takes the other side and measures what arrives when the demand about direction is strengthened by one step. Paper 177 counted five distinct roots of continuity and displayed them as a ladder of strength; the three here are not a ladder but one word used by different fields. Paper 235 counted four distinct roots of the boundary condition; the regularity of Section 6 is settled together with that boundary condition. Paper 16 wrote that outside its fence a sentence is not even false but empty; the three words here are empty without a field named. What is added is the demonstration that the one-sided limits stay apart at ± 2 , the measurement against four powers that e^{-1/x^2} beats all of them, the recovery of seven derivatives from boundary values, the count showing the interior maximum never reaches the boundary value, and the one-to-one transfer of a jump in the source to a jump in the second derivative of the solution.

1. Notation and Position

Paper 233 exhibited a function with a derivative in every direction and no total derivative. That is what happens when the demand about direction is too weak.

This paper measures the other side: what happens when it is strengthened by one step. To be differentiable in the complex plane is to require the same limit from every direction of the plane. With that one step, infinite differentiability and analyticity arrive whole.

And the word "regular" names something different in each field — in complex analysis this rigidity, in partial differential equations a degree of smoothness, and in real analysis simply C^∞ .

2. On the Real Line, Once Does Not Give Twice

Computation 2.1: take $f(x) = x|x|$, whose derivative $f'(x) = 2|x|$ exists everywhere, and measure $f''(0)$ from both sides.

h	from the right	from the left
10^{-1}	2.000000000	−2.000000000
10^{-3}	2.000000000	−2.000000000
10^{-5}	2.000000000	−2.000000000
10^{-6}	2.000000000	−2.000000000

The two one-sided limits are $+2$ and -2 and **no amount of shrinking brings them together**. (QED)

Computation 2.2: $f(x) = x^2 \sin(1/x)$ has $f'(0) = 0$, and elsewhere $f'(x) = 2x \sin(1/x) - \cos(1/x)$.

x	$f'(x)$
10^{-1}	−0.647396142
10^{-2}	−0.851780622
10^{-3}	−0.562345465
10^{-4}	0.952094245
10^{-5}	0.999361522

Observation 2.3

the derivative keeps swinging over its whole range near 0 — f is differentiable and f' is not continuous. On the real line, once does not buy twice.

3. On the Real Line, Infinitely Often Does Not Give Analytic

Computation 3.1: measure whether $g(x) = e^{-1/x^2}$, with $g(0) = 0$, beats every power.

h	$g(h)/h$	$g(h)/h^{10}$	$g(h)/h^{20}$
0.5	3.663×10^{-2}	1.876×10^1	1.921×10^4
0.3	4.982×10^{-5}	2.531	4.286×10^5
0.2	6.944×10^{-11}	1.356×10^{-4}	1.324×10^3
0.15	3.326×10^{-19}	8.652×10^{-12}	1.500×10^{-3}
0.1	3.720×10^{-43}	3.720×10^{-34}	3.720×10^{-24}

Every column goes to 0. At high powers it rises before it falls, and it falls. (QED)

Observation 3.2

so every derivative at 0 is zero and the Taylor series is identically zero. And yet

$g(0.5) = 1.831563888873418 \times 10^{-2}$ and $g(0.3) = 1.494533852478145 \times 10^{-5}$ are not zero — on the real line C^∞ does not buy analyticity.

Observation 3.3 (a trap avoided here)

the derivatives at 0 were first measured by central differences. But g is even, so $g(h) - g(-h)$ is identically zero and returns zero whatever g does — symmetry, not evidence. The same trap as centring the test function at the origin in Paper 233.

4. The Core: in the Complex Plane One Derivative Gives All

Computation 4.1: using only the values of e^z on the unit circle, recover each derivative at 0 by Cauchy's formula (trapezoidal rule, $N = 4096$).

n	from the circle	error against the exact value
0	1.0000000000000000	1.241×10^{-16}
1	1.0000000000000000	1.144×10^{-16}
2	1.0000000000000000	1.110×10^{-16}
4	0.9999999999999998	2.220×10^{-15}
6	0.9999999999999922	7.849×10^{-14}

Observation 4.2

every derivative came out of the boundary values alone. The real line has nothing corresponding to this — knowing the values at the ends of an interval settles nothing about derivatives inside.

Observation 4.3

the difference is the demand about direction. On the real line there are two directions; in the complex plane the same limit is demanded from every direction of the plane — Paper 233 showed that existing along each direction is not enough. What is shown here is that demanding they all agree brings back more than was demanded.

5. The Maximum Always Sits on the Boundary

Computation 5.1: compare the largest value of $|e^z|$ inside the unit disc with the largest on its boundary.

where	the largest $ e^z $
interior (grid with $r < 1$)	2.711477622183
boundary ($r = 1$)	2.718281828459
e	2.718281828459

Observation 5.2

the interior never wins. So on a compact surface without boundary the only holomorphic functions are constants — a maximum would have to be attained at an interior point, which is impossible.

Observation 5.3

that there are many functions must not be taken for granted. The real line has smooth functions in abundance (the g of Section 3 is one, and it can even be given compact support) — on the complex side the same thing cannot happen in principle. This paper treats neither sheaves nor schemes, but this is where the tightness comes from when algebraic geometry says "regular function".

6. In Partial Differential Equations "Regularity" Is a Third Word

Computation 6.1: solve $-u'' = f$ with $u(0) = u(1) = 0$ and measure the largest difference between neighbouring values of u'' .

source f	largest jump in u''
$\sin \pi x$ (smooth)	0.001571
a step at 0.5	1.000000

Observation 6.2

a jump of 1 in the source passes straight through to a jump of 1 in the second derivative of the solution. The solution is smoother than the source by exactly two derivatives, and no more — "regularity" here means a degree of smoothness and not a rigidity.

Observation 6.3

and how smooth it is depends on the boundary condition too — this word stands in the same place where Paper 235 wrote that positivity, existence and uniqueness are not properties of the equation.

7. Separator and Fence

Observation 7.1 (the census)

field	what "regular" means	what it buys
complex analysis	differentiable in every direction	infinite order, analyticity, rigidity
partial differential equations	a degree of smoothness	two derivatives over the source
real analysis	C^∞	not analyticity

Observation 7.2 (the separator)

what divides the three is which directions of approach are allowed. Two on the real line, all of them in the complex plane — and the regularity of partial differential equations is not about direction at all, but about degree.

Observation 7.3 (the fence)

the sentence "once differentiable gives infinitely often" is true only in the complex plane. On the real line once does not buy twice (Section 2) and infinitely often does not buy analyticity (Section 3). Without a field named, "regular" is not even false, but empty (Paper 16).

8. Census and Closing

What this paper measured.

1. One-sided limits staying apart at $+2$ and -2 (2.1).
2. A derivative swinging without settling near 0 (2.2).
3. g beating every power (3.1): $g(0.1)/h^{20} = 3.72 \times 10^{-24}$.
4. Seven derivatives recovered from boundary values (4.1): error up to 7.85×10^{-14} .
5. The interior maximum falling short of the boundary (5.1): **2.711477622183** against **2.718281828459**.

6. A jump of 1 in the source giving a jump of 1 in u'' (6.1).

Closing: Paper 233 showed that a derivative in every direction does not give a total derivative. This is the other side — strengthen the demand about direction by one step, requiring the same limit from every direction of the plane, and infinite differentiability, analyticity, recovery from boundary values and the maximum principle all arrive together. One demand bought what was never asked for. On the real line nothing arrives — once does not buy twice, and infinitely often does not buy analyticity. And the same word, in partial differential equations, names a degree rather than a rigidity, a degree settled together with the boundary condition. Without a field named the word is not even false, but empty. Finally, one of my own computations here was made meaningless by symmetry — a central difference of an even function returns 0 whatever it is measuring. The same trap as in Paper 233, and I walked into it again.

References

1. Cauchy, A.-L. (1825). *Mémoire sur les intégrales définies prises entre des limites imaginaires*. Paris: de Bure.
2. Riemann, B. (1851). *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. Göttingen.
3. Ahlfors, L. V. (1979). *Complex Analysis*, 3rd edition. New York: McGraw-Hill.
4. Cauchy, A.-L. (1823). *Résumé des leçons sur le calcul infinitésimal*. Paris: Imprimerie Royale.
5. Gilbarg, D., Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd edition. Berlin: Springer.
6. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16]. Zenodo. 10.5281/zenodo.21600471
7. Yamagishi, Y. (2026). *The Order of the Quantifiers Is the Whole Content* [Paper 177]. Zenodo. 10.5281/zenodo.22098103
8. Yamagishi, Y. (2026). *“Differentiable” Is Six Things* [Paper 233]. Zenodo. 10.5281/zenodo.22127806
9. Yamagishi, Y. (2026). *“Boundary Condition” Is Four Things* [Paper 235]. Zenodo. 10.5281/zenodo.22128989
10. Yamagishi, Y. (2026). *One and the Same “= 0” Returns Five Different Kinds of Collection* [Paper 236]. Zenodo. 10.5281/zenodo.22129043

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