

“Diverges” Is Four Things, and One of Them Converges in Another Metric

— Cesaro, Abel, Borel and the 2-adic Distance: the Separator Is Which Metric Measures It and What Is Allowed

Besides Partial Sums — [Paper 239]

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— *From thinking to syncing.* —

ABSTRACT

One word, "diverges", names at least four different states. This paper separates them in numbers and shows that the last is relative to the choice of metric. **NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED.** First, bounded oscillation. The partial sums of $1 - 1 + 1 - 1 + \dots$ run 1, 0, 1, 0, and the Cesàro average gives 0.500000000000 while the Abel sum reaches 0.50000025: the two agree (Section 2). Second, unbounded oscillation. For $1 - 2 + 3 - 4 + \dots$, $(C, 1)$ sits at 0.500025001 at odd n and at 0 at even n and never settles — $(C, 2)$ gives 0.250139673 and Abel 0.250025001875, both $1/4$ (Section 3). Third, radius of convergence zero. At $x = 0.1$ the Euler series approaches 0.09158192 by the eleventh term and flies out to -0.0195940911 by the twenty-sixth — and its Borel sum 0.091563333939788081876 satisfies the original differential equation to a residual of 8.11×10^{-19} (Section 4). Fourth, and this is the core. The partial sums of $1 + 2 + 4 + 8 + \dots$ run away from -1 in the real metric, $4.0 \rightarrow 2.048 \times 10^3 \rightarrow 2.199 \times 10^{12}$, and converge to -1 in the 2-adic metric, $0.25 \rightarrow 4.88 \times 10^{-4} \rightarrow 4.55 \times 10^{-13}$ (Section 5). Fifth, and this is the fence. On $1 + 2 + 3 + \dots$, Abel gives 999000, $(C, 1)$ gives 6.6677×10^7 and $(C, 2)$ gives 2.2229×10^7 : all three run to infinity — so $-1/12$ is not what any summation method returns (Section 6).

Scope of this paper (scope note): NO NEW MATHEMATICAL THEOREM AND NO NEW LAW IS CLAIMED — Cesàro summation, Abel summation, Borel summation, Frobenius's theorem, the p -adic distance, the Euler series as a formal solution, and the divergence of perturbation series are all standard. No general theory of summation is built: regularity, linearity and stability are quoted and not proved. No Tauberian theorem is treated; the converse implications are not entered. No p -adic analysis is entered; the 2-adic distance appears only as a witness. No history is written, and no claim is made that drawing a distinction is what solved anything — such stories always look that way in hindsight, and this paper offers a census only, with no claim that a census solves anything. Quantum field theory is not discussed; the divergence of perturbation series is left to the references. Relation to earlier papers: Paper 211 showed that an asymptotic expansion is best at the $1/x$ -th term; this paper takes the other road, not truncation but summation. Paper 232 placed $1 + 2 + 3 + \dots = -1/12$ as a convention reached by analytic continuation; this paper supplies the check, showing in numbers that three summation methods actually fail on it. Paper 234 counted the ultrametric; the 2-adic distance of Section 5 uses that tool directly. Paper 197 counted four distinct roots of "converges"; this is its mirror image. What is added is the four divergences set out in one format, the two-sided measurement showing $(C, 1)$ holds two values by parity, the residual check that the Borel sum satisfies the original equation, the distance of one sequence to -1 under two metrics side by side, and the numerical demonstration that three summation methods diverge on $1 + 2 + 3 + \dots$.

1. Notation and Position

Definition 1.1

a series $\sum a_n$ diverges when the sequence of partial sums s_n does not converge.

Two things are not written in that definition — in which metric convergence is measured, and whether anything besides partial sums is allowed. This paper varies both.

The three conditions on a summation method (quoted here, not proved):

1. **Regularity:** on a convergent series it returns the original value.
2. **Linearity:** it respects sums and scalar multiples.
3. **Stability:** removing the first term changes the value by exactly that term.

Because of these three, arbitrary values cannot be assigned. Section 6 is the demonstration.

Position: Paper 211 treated where to truncate; this paper takes the side that does not truncate but admits another operation.

2. The First Kind: Bounded Oscillation

Computation 2.1: partial sums and Cesàro averages of $1 - 1 + 1 - 1 + \dots$.

terms	partial sum	Cesàro average
1	1	1.000000000000
2	0	0.500000000000
3	1	0.666666666667
10	0	0.500000000000
20000	0	0.500000000000

Computation 2.2: the Abel sum, the limit of $1/(1+x)$.

x	$1/(1+x)$
0.9	0.526315789474
0.99	0.502512562814
0.999	0.500250125063
0.999999	0.50000025

Observation 2.3

the two methods returned the same $1/2$. Not by accident — Cesàro summable implies Abel summable with the same value (Frobenius). This paper quotes and does not prove it.

3. The Second Kind: Unbounded Oscillation

Computation 3.1: measure $1 - 2 + 3 - 4 + \dots$.

terms	partial sum	$(C, 1)$	$(C, 2)$
9	5	0.5555555556	0.377072310
10	−5	0.000000000	0.339365079
999	500	0.500500501	0.252296826
1000	−500	0.000000000	0.252044530
19999	10000	0.500025001	0.250152181
20000	−10000	0.000000000	0.250139673

$(C, 1)$ holds $1/2$ at odd n and 0 at even n forever, and therefore **does not converge**. (QED)

Computation 3.2: the Abel sum, the limit of $1/(1+x)^2$.

x	$1/(1+x)^2$
0.9	0.277008310249
0.99	0.252518875786
0.9999	0.250025001875

Observation 3.3

one series is not summable by one method and equal to $1/4$ by two others. "A divergent series has a sum" means nothing until the method is named — in the language of Paper 16, outside its fence it is empty.

Observation 3.4 (a trap avoided here)

$(C, 1)$ was first tabulated at even n only. The column then reads 0.000000000 throughout, which looks like convergence to 0 — an oscillating quantity must be read at both phases. The same kind of trap as the central difference of an even function in Paper 238.

4. The Third Kind: Radius of Convergence Zero

Computation 4.1: sum the Euler series $\sum_{n \geq 0} (-1)^n n! x^{n+1}$, the formal solution of $x^2 y' + y = x$, at $x = 0.1$.

terms	partial sum	size of the last term
1	0.1	0.1
6	0.09152	0.00012
11	0.09158192	3.6288×10^{-5}
16	0.0914840440064	0.00013076744
21	0.09319427285184512	0.002432902
26	-0.01959409108642064	0.1551121

The series diverges at every $x \neq 0$, the terms growing like $n!$. (QED)

Computation 4.2: compute the Borel sum $x \int_0^\infty e^{-t} / (1 + xt) dt$ and put it back into the equation.

quantity	value
the Borel sum	0.091563333939788081876
residual of $x^2 y' + y - x$	8.11×10^{-19}

Observation 4.3

a divergent series acquired one meaningful value, and that value actually satisfies the equation the series was the formal solution of — "divergent, therefore meaningless" is false. Where best to truncate was the subject of Paper 211; this is the road that does not truncate.

5. The Core: Divergence Is Relative to a Metric

Computation 5.1: the partial sums of $1 + 2 + 4 + 8 + \dots$ are $2^{n+1} - 1$. Measure their distance to -1 in two metrics.

n	partial sum	real distance	2-adic distance
1	3	4.000	0.25
3	15	1.600×10^1	0.0625
10	2047	2.048×10^3	4.883×10^{-4}
20	2097151	2.097×10^6	4.768×10^{-7}
40	2199023255551	2.199×10^{12}	4.547×10^{-13}

Observation 5.2

one and the same sequence runs away under one metric and converges geometrically to -1 under the other. In the 2-adic metric $|s_n - (-1)|_2 = 2^{-(n+1)}$ exactly — so the sentence "this series diverges" has the fence "in which metric" omitted.

Observation 5.3

the 2-adic distance used here is the ultrametric counted in Paper 234. There it made every triangle isosceles — here it turns divergence into convergence. This paper does not enter p -adic analysis; the distance appears only as a witness.

Observation 5.4

and both answers are correct. Writing -1 on the real side is false, and writing "divergent" on the 2-adic side is false — the two are not competing. They answer different questions.

6. The Fence: Arbitrary Values Cannot Be Assigned

Computation 6.1: apply the methods of Sections 2 and 3 to $1 + 2 + 3 + \dots$.

terms	partial sum	$(C, 1)$	$(C, 2)$
10	5.5000×10^1	2.2000×10^1	9.5000
100	5.0500×10^3	1.7170×10^3	5.8950×10^2
1000	5.0050×10^5	1.6717×10^5	5.5890×10^4
20000	2.0001×10^8	6.6677×10^7	2.2229×10^7

Computation 6.2: the Abel sum, $x/(1 - x)^2$.

x	$x/(1 - x)^2$
0.9	90
0.99	9900
0.999	999000

Observation 6.3

all three run to infinity. So $-1/12$ is not what any of these methods returns — "one may assign a divergent series whatever value one likes" is false.

Observation 6.4

$-1/12$ arrives from a different mechanism, analytic continuation — which is what Paper 232 placed as a convention. What this paper adds is the number showing that the summation side genuinely fails.

7. Separator and Fence

Observation 7.1 (the census)

kind of divergence	example	what gives it a value
bounded oscillation	$1 - 1 + 1 - 1$	an average (Cesàro, Abel: $1/2$)
unbounded oscillation	$1 - 2 + 3 - 4$	two averages ($(C, 2)$, Abel: $1/4$)
radius of convergence zero	the Euler series	an integral transform (Borel)
divergent in the real metric	$1 + 2 + 4 + 8$	another metric (2-adic: -1)
none of them work	$1 + 2 + 3 + \dots$	no summation method does

Observation 7.2 (the separator)

there are two — which metric measures it, and what is allowed besides partial sums (averages, weights, integral transforms). The symbol $\sum a_n$ writes neither — the same shape as f' in Paper 233, $d(x, y)$ in Paper 234, $-u''$ in Paper 235 and $"= 0"$ in Paper 236.

Observation 7.3 (the fence)

there are three. First, regularity, linearity and stability, which is why arbitrary values cannot be assigned (Section 6). Second, by which method: $1 - 2 + 3 - 4$ is not summable by $(C, 1)$ and equals $1/4$ by $(C, 2)$ (Section 3). Third, in which metric: $1 + 2 + 4 + 8$ diverges in the reals and converges 2-adically (Section 5).

Observation 7.4 (what this paper does not do)

no claim is made that drawing a distinction is what solved anything. Such stories always look that way in hindsight — what was done here is a census, and no claim is made that a census solves anything.

8. Census and Closing

What this paper measured.

1. Cesàro and Abel agreeing (2.1, 2.2): 0.500000000000 and 0.50000025 .
2. $(C, 1)$ holding 0 and $1/2$ by parity (3.1): no convergence.
3. The Borel sum satisfying the equation (4.2): residual 8.11×10^{-19} .

4. One sequence pointing opposite ways in two metrics (5.1): 2.199×10^{12} against 4.547×10^{-13} .

5. Three summation methods failing on $1 + 2 + 3 + \dots$ (6.1, 6.2).

Closing: one word, "diverges", named four different states. Bounded oscillation yields to an average, unbounded oscillation needs two, and a radius of convergence of zero needs an integral transform. And the fourth was missing not an operation but a metric — $1 + 2 + 4 + 8 + \dots$ runs away in the real distance and converges geometrically to -1 in the 2-adic one. The two answers are not competing; they answer different questions. And there is a fence — on $1 + 2 + 3 + \dots$, Abel and $(C, 1)$ and $(C, 2)$ all run to infinity. $-1/12$ is not what any summation method returns: it comes from analytic continuation, a different mechanism (Paper 232). "One may assign a divergent series whatever value one likes" is false. Finally, this paper walked into a trap once as well — $(C, 1)$ was tabulated at even n only, and a column of zeros appeared. An oscillating quantity must be read at both phases — the same kind of trap as the central difference of an even function in Paper 238.

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