

Bounded by a Sum Is Not One Property

— Entropy, Norm, Rank, Measure, Dimension: The Equality Condition Differs in All Five, and One Is Bounded by a Maximum Rather Than a Sum — [Paper 243]

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— *From thinking to syncing.* —

Abstract

Paper 241 asked whether the whole is a function of the parts and listed the quantities for which it is not. This paper asks the neighbouring question: does the same inequality $f(A * B) \leq f(A) + f(B)$ state the same property? No new mathematical theorem and no new law is claimed. First, five quantities share the shape: entropy, norm, matrix rank, measure and the dimension of a sum of subspaces (Section 2). Second, the composition differs in all five: tensor product, vector sum, matrix sum, union, subspace sum (Section 3). Third, the equality condition differs in all five. Entropy closes the gap at a product state and opens it to **2.000000000** on a Bell state (Section 4). Fourth, when the target is bounded the sum is cut off before it is reached. With $\dim U = 6$ and $\dim V = 7$ the sum is **10** and never the **13** of the bound (Section 5). Fifth, this is the core. One of the five is bounded by a maximum and not a sum. The exponent of the union of a Cantor set with an interval is **0.996526** at $k = 12$. It goes to the maximum **1**, not to the sum **1.630930** (Section 6). Sixth, one side falls in the quantum case and not in the classical one. Two perfectly correlated bits give $H(XY) = 1$, never below $H(X)$, while a Bell state gives $S(AB) = 0$, below $S(A) = 1$ (Section 7). Seventh, the first measurement failed. Sampling by points returned **0.961077** for the union, an impossible value below that of one of its own parts (Section 8).

Scope of this paper (scope note) : No new mathematical theorem and no new law is claimed: subadditivity of von Neumann entropy, the triangle inequality, subadditivity of rank, measure and dimension, and the fact that Hausdorff dimension takes a maximum over countable unions are all standard. No measured value is cited: every number is computed from a definition or counted with a pseudorandom sample. No category theory is introduced: the existence of a categorical name for this shape is mentioned, and nothing is derived from that name. No theory of entanglement is built: Sections 4 and 7 place two states side by side. No fractal geometry is developed: Section 6 uses only a direct count of intervals. It is not claimed that the five are really the same thing, nor that they are really different. What is shown is the formal fact that the composition and the equality condition differ. Relation to earlier papers: Paper 241 asked whether the whole is a function of the parts. This paper counts the cases where it is at least bounded above. Paper 227 separated the cases where an inequality touches equality from those where it does not: Section 4 lines up five touching conditions and shows they all differ. Paper 94 showed that dimension is not one thing: Section 6 measures the one that follows a maximum rather than a sum. Paper 162 showed that the whole can be less known than the part: Section 7 is the same fact on the side of the inequality. Paper 206 recorded a failure in measuring a box dimension: Section 8 is the same trap in a different form. What is added is the uniform presentation of the five, the separation of composition from equality condition, one example whose target monoid is not addition, and the record of a sampling method returning an impossible value.

1. Notation and placement

Setting: for a quantity f and an operation $*$ that combines two things, consider the shape $f(A * B) \leq f(A) + f(B)$.

This shape appears throughout.

1. von Neumann entropy: $S(AB) \leq S(A) + S(B)$, composition by tensor product
2. norm: $\|x + y\| \leq \|x\| + \|y\|$, composition by vector sum
3. matrix rank: $\text{rank}(A + B) \leq \text{rank} A + \text{rank} B$, composition by matrix sum
4. measure: $\mu(A \cup B) \leq \mu(A) + \mu(B)$, composition by union
5. dimension: $\dim(U + V) \leq \dim U + \dim V$, composition by subspace sum

On paper the five carry the same shape. This paper counts what is the same and what is not.

Category theory has a name for this shape. It is mentioned here and nothing is derived from it.

2. The composition differs in all five

Look at $*$ first. The symbol reads as a plus or a union, but what is being combined differs.

quantity	composition *	where $A * B$ lives
entropy	tensor product	the state space of the joint system
norm	vector sum	the same vector space
rank	matrix sum	the same matrix space
measure	union of sets	the same measurable space
dimension	sum of subspaces	the same ambient space

Observation 2.1

Only the tensor product changes the place where the result lives. In the other four, A , B and $A * B$ share one place, so calling them the same inequality already compares different types.

Observation 2.2

Paper 241 counted eight symbols that look like a plus. What this paper looks at is not that symbol but the inequality sign: the next shelf in the same system.

3. Equality is touched under five different conditions

Check 3.1: for entropy, measure the case that touches equality and the case that is furthest from it, in bits.

state	$S(A)$	$S(B)$	$S(AB)$	gap
Bell state (entangled)	1.000000000	1.000000000	0.000000000	2.000000000
maximally mixed product	1.000000000	1.000000000	2.000000000	0.000000000

Check 3.2: for the norm, take parallel vectors $y = x$. Over 100000 draws the largest value of $|2\|x\| - \|2x\||$ is 0.000×10^0 , an exact equality. For two independent vectors the mean gap is 0.733737 at $d = 2$, 1.806171 at $d = 10$ and 5.840427 at $d = 100$ over 200000 draws each, with no violations.

Check 3.3: measure the touching condition for the other three.

quantity	equality holds when	measured
entropy	the state is a product	gap 0.000000000
norm	the vectors are parallel	gap 0.000×10^0
rank	the images intersect trivially	rank = 3 = 3 + 0
measure	the overlap is null	$ A = B = 1$, union 2.0, overlap 0.0
dimension	$U \cap V = \{0\}$	dim $U = 3$, dim $V = 4$, sum 7

Observation 3.4

All five can touch equality, and the condition differs in all five. Being a product, being parallel, meeting trivially, not overlapping, being independent: none of the five follows from any other.

Observation 3.5

Paper 227 separated inequalities that reach equality from those that do not. Here all five reach it: but they reach it in different places.

4. A bounded target cuts the sum off before it is reached

Check 4.1: measure the dimension of a sum of subspaces inside a 10-dimensional space.

$\dim U$	$\dim V$	sum	$\dim(U + V)$	ambient
3	4	7	7	10
6	7	13	10	10
5	5	10	10	10

Observation 4.2

In the second row the bound of 13 is never approached. What holds the value down is not subadditivity but the ambient dimension 10: the inequality holds and a different ceiling is doing the work.

Observation 4.3

Writing that $f(A * B) \leq f(A) + f(B)$ holds does not say whether it is the binding constraint. Another ceiling may be lower.

5. The core: one is bounded by a maximum, not a sum

Setting: let the composition be union and the quantity be the box dimension. Take the **Cantor set** and the **interval [2, 3]** and, at scale 3^{-k} , count the meeting intervals directly.

k	Cantor set	interval [2, 3]	union	exponent of union
4	16	81	97	—
6	64	729	793	0.963991
8	256	6561	6817	0.983067
10	1024	59049	60073	0.992275
12	4096	531441	535537	0.996526

The exponent of the Cantor set is $\log 2 / \log 3 = 0.630930$ at every step, and that of the interval is 1.

Observation 5.1

The exponent of the union goes to 1. It does not go to the sum $0.630930 + 1 = 1.630930$. Here the target is not $(\mathbb{R}, +)$ but $(\mathbb{R}, \max)$: the phrase bounded by a sum does not fit at all.

Observation 5.2

Four of the five are bounded by a sum and one by a maximum. The separator is what the target calls addition: the inequality has the same shape while the operation on the right differs.

Observation 5.3

Paper 94 showed that dimension is not one thing. This paper sees that one of them lives in a different algebra from the other four quantities.

6. What falls in the quantum case and not in the classical one

Check 6.1: measure whether the whole can fall below a part, in bits.

system	part	whole	does the whole fall below
two perfectly correlated bits (classical)	$H(X) = 1.000000$	$H(XY) = 1.000000$	no
Bell state (quantum)	$S(A) = 1.000000$	$S(AB) = 0.000000$	**yes**

Observation 6.2

Subadditivity holds classically and quantum mechanically alike. What fails in the quantum case is the lower bound: that one inequality holds says nothing about the one next to it.

Observation 6.3

Paper 162 wrote that the whole can be less known than the part. This paper sees the same fact from the direction of the inequality sign.

7. The first measurement failed

Record: Section 5 was first measured by **sampling points**. The Cantor set was sampled with 2^{14} points and the interval with **200000**, and the boxes were counted on a grid.

object	exponent from sampling	correct value
Cantor set	0.676042	0.630930
interval	1.000000	1.000000
union	0.961077	1.000000

Observation 7.1

The exponent of the union came out below that of the interval it contains. That is impossible: a union contains its parts. The number landed outside the possible range, which is how the instrument was found to be wrong.

Observation 7.2

Cause: a sample has a fixed number of points. At fine scales the box count saturates at the number of points: the slope is pushed down artificially. Paper 206 fell into the same kind of trap while measuring a box dimension. There it was the placement of the grid, here the finiteness of the sample.

Observation 7.3

Discipline: when the object saturates, a quantity measured in a fine limit measures the saturation instead. If a check lands in a logically impossible range, suspect the instrument and not the conclusion.

8. Separator and fences

Separator: there are three.

1. what the composition $*$ is (tensor product, sum, union)
2. what condition touches equality (product, parallel, trivial intersection, no overlap, independence)
3. what operation the target carries (+ or **max**)

Fences:

1. It is not claimed that the five are really the same.

2. Nor that they are really different: only that the composition and the equality condition differ.
3. Nothing is derived from the categorical name. It is only mentioned.
4. No physical meaning is drawn from subadditivity.
5. Section 6 places two states side by side and treats no measure or classification of entanglement.

9. Roll call and closing

1. Five share the shape (Section 2): the composition differs in all five.
2. The equality condition differs in all five (Checks 3.1 to 3.3): gaps 2.000000000 and 0.
3. Another ceiling can bind first (Check 4.1): $6 + 7$ gives 10.
4. One is bounded by a maximum (Section 5): 0.996526, where a sum would give 1.630930.
5. The lower bound fails in the quantum case (Check 6.1): $H(XY) = 1$ against $S(AB) = 0$.
6. Sampling returned an impossible value (Section 7): 0.961077.

Closing: bounded by a sum was not one property. The composition differs, the touching condition differs, and the binding ceiling differs. And one of the five is not bounded by a sum at all. It is bounded by a maximum. That two inequalities share a shape does not mean the operation on the right is the same. Category theory has a name for this shape. It was not borrowed here: writing down the composition and the equality condition was enough.

Revision Record

Second edition (September 2026): no claim, theorem or check has been altered from the first edition. In the English references, the lines citing earlier papers of this corpus still carried their Japanese titles; they now carry the English titles. Nothing of substance has changed, and nothing has been withdrawn.

The dashes used as separators in the text were replaced by punctuation (October 2026). No claim, number or reference was changed.

References

1. Araki, H., Lieb, E. H. (1970). Entropy inequalities. *Communications in Mathematical Physics*, 18.
2. Nielsen, M. A., Chuang, I. L. (2000). *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press.
3. Falconer, K. (2014). *Fractal Geometry: Mathematical Foundations and Applications*, 3rd edition. Chichester: Wiley.
4. Mac Lane, S. (1998). *Categories for the Working Mathematician*, 2nd edition. New York: Springer.

5. Cover, T. M., Thomas, J. A. (2006). *Elements of Information Theory*, 2nd edition. Hoboken: Wiley.
6. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16]. Zenodo. 10.5281/zenodo.21600471
7. Yamagishi, Y. (2026). *“Dimension” Is Not One Thing* [Paper 94]. Zenodo. 10.5281/zenodo.21758581
8. Yamagishi, Y. (2026). *The Whole Can Be Better Known Than Its Parts* [Paper 162]. Zenodo. 10.5281/zenodo.22092568
9. Yamagishi, Y. (2026). *The Spaces Converge and the Dimension Does Not* [Paper 206]. Zenodo. 10.5281/zenodo.22110637
10. Yamagishi, Y. (2026). *The Same $\leq$ Points at a Wall You Can Touch and at One You Cannot* [Paper 227]. Zenodo. 10.5281/zenodo.22123129
11. Yamagishi, Y. (2026). *Before Asking Whether It Can Be Made into Addition, Ask Whether There Is a Binary Operation at All* [Paper 241]. Zenodo. 10.5281/zenodo.22130189

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