

To Determine an Object You Need Every Other One

— What the Yoneda Lemma Requires and What Is Lost When the Probe Is Weakened: Counted over the 156

Graphs on Six Vertices — [Paper 244]

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— *From thinking to syncing.* —

ABSTRACT

The Yoneda lemma says that an object is completely determined by its relations to every other object. This paper asks about that every—where does the separating power fall when the probe is weakened? No new mathematical theorem and no new law is claimed. First, the graphs on six vertices fall from **32768** labelled objects to **156** isomorphism classes (Section 3). Second, the degree sequence is not enough. It takes only **102** distinct values and **84** classes survive unseparated in **30** groups (Section 3). Third, the spectrum is not enough either. It takes **151** values and **10** classes survive in **5** groups (Section 3). Fourth, this is the core. Adding the triangle count leaves **151** unchanged—the probe was strengthened and the separating power did not rise at all (Section 4). Fifth, the same thing happens for groups. D_4 and Q_8 share one character table, yet $\text{Hom}(\mathbb{Z}/2, -)$ separates them at **6** against **2**—while $\text{Hom}(\mathbb{Z}/4, -)$ does not, at **8** against **8** (Section 5). Sixth, Yoneda does not justify the method of this body of work. The lemma is a theorem inside a category, and taking it as a metaphor drops to the third layer (Section 6).

Scope of this paper (scope note) : No new mathematical theorem and no new law is claimed—the Yoneda lemma, the existence of cospectral non-isomorphic graphs, and the coincidence of the character tables of D_4 and Q_8 are all standard. No measured value is cited—every number is obtained by exhaustive enumeration. No category theory is developed—only the statement of the lemma is used, and neither its proof nor any generalisation is treated. Nothing is said about the complexity of graph isomorphism—Section 3 is an exhaustive count in the finite case of six vertices. No representation theory is developed—Section 5 quotes the character tables as known. It is not claimed that the Yoneda lemma justifies the method of this body of work—Section 6 exists precisely to stop that claim from being made. Relation to earlier papers: Paper 152 showed that two groups can share a character table without being isomorphic—this paper counts that phenomenon as a function of the strength of the probe. Paper 153 separated three stages of forgetting and exhibited the absence of a right adjoint—this paper treats the neighbouring theorem on the same shelf. Paper 102 treated the question whether one can hear the shape of a drum—Section 3 is its finite version, counted exhaustively. Paper 163 showed that there exists and here it is are different sentences—Section 6 says that determined and computable are different sentences. Paper 246 puts the same instrument, a count of Hom sets, to a different use—it asks whether homomorphisms out of a free group number a power of the order, and so checks whether an adjunction is present. Here the same instrument measures separating power—one instrument, two quantities. What is added is the exhaustive count over three strengths of probe, a numerical case where a stronger probe adds nothing, the confirmation in groups by counting Hom sets, and an explicit fence against using Yoneda as a metaphor.

1. Notation and placement

The Yoneda lemma says roughly this. An object X is determined up to isomorphism by the collection of maps $\text{Hom}(Y, X)$ from every other object Y .

The work is done by the word every. This paper counts what happens when it is weakened.

Setting: take a collection of objects and a **probe**, a function that is applied to an object and returns a value. **When a probe returns the same value on two objects it does not separate them.**

The lemma is not proved here. Only its statement is used.

2. Three strengths of probe

Material: the simple graphs on 6 vertices, of which there are $2^{15} = 32768$ labelled ones.

Three probes are used: the **degree sequence**, the **spectrum** of the adjacency matrix, and **the spectrum together with the triangle count**. For comparison the **isomorphism type itself** is included.

Observation 2.1

The three grow stronger in order. The spectrum determines the degree sum, and the triangle count is $\text{tr}(A^3)/6$, also a function of the spectrum—so the third probe should add nothing to the second. This paper checks that with numbers.

3. Check: separating 156 classes with three probes

Check 3.1: apply all **720** vertex permutations to each of the **32768** graphs, take a canonical form and count isomorphism classes.

counted	number
labelled graphs	32768
isomorphism classes	156

Check 3.2: separate the **156** classes with the three probes.

probe	distinct values	classes not separated	groups
degree sequence	102	84	30
spectrum	151	10	5
spectrum with triangle count	151	10	5
isomorphism type	156	0	0

Check 3.3: extract the smallest pair the spectrum cannot separate.

graph	degree sequence	edges	triangles
<i>A</i>	0, 1, 1, 1, 1, 4	4	0
<i>B</i>	0, 0, 2, 2, 2, 2	4	0

The shared spectrum is $-2, 0, 0, 0, 0, 2$.

Observation 3.4

The degree sequence separates this pair, and the spectrum does not. A weaker probe separates where a stronger one fails——strength is not a total order on separating power.

4. The core: a stronger probe need not separate more

Adding the triangle count left the distinct values at 151 and the unseparated classes at 10.

Observation 4.1

The triangle count is a function of the spectrum, $\text{tr}(A^3)/6$. Adding it therefore adds no information, and the numbers did not move——the count confirms what was expected.

Observation 4.2

This is the core. Whether more probes separate more depends on whether the new one can be recovered from the old. Measure more invariants and you will eventually tell them apart is false——dependent invariants advance nothing.

Observation 4.3

What the Yoneda lemma demands is exhausting the objects that are independent in this sense. Finitely many probes can be insufficient in principle——10 of the 156 classes hid inside 151 values.

5. The same happens for groups

Check 5.1: take D_4 , the symmetries of a square, and Q_8 , the quaternion group. They share a character table and are not isomorphic (Paper 152). Count $|\text{Hom}(\mathbb{Z}/n, G)|$, the number of g with $g^n = e$.

group	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\mathbb{Z}/6$	1	2	3	2
S_3	1	4	3	4
D_4	1	6	1	8
Q_8	1	2	1	8

Observation 5.2

$\text{Hom}(\mathbb{Z}/2, -)$ gives 6 against 2 and separates them. The character table failed, and counting maps from a single object succeeded.

Observation 5.3

Yet $\text{Hom}(\mathbb{Z}/4, -)$ gives 8 against 8 and does not separate. The same shape of probe changes its power when the test object changes—which object works cannot be known in advance. That is why Yoneda demands every object.

Observation 5.4

The same instrument measures a different quantity in the neighbouring paper. Paper 246 counts whether homomorphisms out of a free group number exactly the order raised to the n , and thereby checks whether an adjunction is present. What this section measured with the same count is separating power—One instrument, a count of Hom , answering does it have a partner on one side and can it tell them apart on the other.

6. Yoneda does not justify the method

This is a fence. The lemma reads as saying that a thing is determined by its relations. The method of this body of work—writing what a thing separates rather than what it is—has a similar shape.

Observation 6.1

Similarity is not justification. The Yoneda lemma is a theorem inside a category, about Hom sets and natural transformations. A methodology is not such an object——so the lemma says nothing about it.

Observation 6.2

Separate two readings. (a) Yoneda as a theorem: the Hom functor is fully faithful. It is proved. (b) Yoneda as a metaphor: things are determined by relations. It is not proved. Speaking (b) with the authority of (a) is the accident this body of work has spent its time avoiding.

Observation 6.3

Paper 163 showed that there exists and here it is are different sentences. The same distinction is needed here——determined and computable are different sentences. Yoneda says X is determined. It gives no way to find it.

7. Separator and fences

Separator: whether the new probe can be recovered from the existing ones. If it can, adding it changes nothing. If it cannot, it may separate more.

Fences:

1. Stronger probe is not a total order on separating power——a pair exists that the degree sequence separates and the spectrum does not.
2. It is not claimed that finitely many probes suffice——10 classes already survive at six vertices.
3. The Yoneda lemma does not justify the method of this work.
4. Determined does not mean computable.
5. Nothing is said about the complexity of graph isomorphism.

8. Roll call and closing

1. Six vertices give 156 isomorphism classes (Check 3.1).

2. The degree sequence leaves **84** (Check 3.2): **102 distinct values**.
3. The spectrum leaves **10** (Check 3.2): **151 distinct values**.
4. Adding triangles leaves **151** (Section 4): **no gain at all**.
5. A weaker probe separates a pair the stronger one misses (Check 3.3).
6. $\text{Hom}(\mathbb{Z}/2, -)$ separates D_4 from Q_8 (Check 5.1): **6 against 2**.
7. $\text{Hom}(\mathbb{Z}/4, -)$ does not (Check 5.1): **8 against 8**.

Closing: the Yoneda lemma demands every object. Replacing that word by a finite list loses something. And what is lost is not governed by the strength of the probe—the degree sequence separated a pair the spectrum missed, and adding triangles gained nothing at all. Measure more and you will know is correct only when what is measured is independent. One fence to close. Yoneda reads as saying that things are determined by relations, and that reading is on the side of metaphor. The theorem speaks about Hom functors and says nothing about methodology.

Revision Record

Second edition (August 2026): no claim, theorem or check has been altered from the first edition. What is added is a single cross-reference. This paper and Paper 246 (Free and Forgetful Come in Pairs) were published on the same day, so at the time of the first edition neither carried a DOI yet and neither could point to the other. The second edition adds **Observation 5.4** and lists Paper 246, with its DOI, among the references. **One and the same instrument, a count of Hom sets, measures separating power here and the existence of an adjoint there. The marks used in the body text for caution and emphasis have also been removed**, their use not having been consistent across the corpus. **Nothing of substance has changed, and nothing has been withdrawn.**

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