

Free and Forgetful Come in Pairs

— A Roll Call of Adjunctions: Counting Hom Sets Checks the Correspondence, and One of Two

Compactifications Is Not Even a Functor — [Paper 246]

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— *From thinking to syncing.* —

ABSTRACT

Operations that add something and operations that throw something away appear again and again—free constructions, completions, localisations, compactifications. These come in pairs—except that not all of them do. No new mathematical theorem and no new law is claimed. First, the pairing can be checked with numbers. Homomorphisms out of a free group number exactly $|G|^n$, which is 8, 64, **512** for D_4 and 8, 64, **512** for Q_8 alike (Section 2). Second, a non-free source gives no power. The count $|\mathrm{Hom}(\mathbb{Z}/2, -)|$ is **6** for D_4 and **2** for Q_8 —neither is a power of 8 (Section 3). Third, this is the core. There are two compactifications, and one is an adjoint while the other is not even a functor. The one-point compactification cannot extend even the plain inclusion $(0, 1) \subset \mathbb{R}$ (Section 4). Fourth, completion is not an adjoint until the class of maps is fixed. With uniformly continuous maps it is one; with merely continuous maps it fails (Section 5). Fifth, the algebraic closure is not an adjoint. It exists and provides no unique map—the same shelf as Paper 179 (Section 6).

Scope of this paper (scope note) : No new mathematical theorem and no new law is claimed—that free functors are left adjoint to forgetful ones, that the Stone–Cech compactification is a left adjoint, that the one-point compactification is functorial only for proper maps, and that the algebraic closure has no universal property are all standard. No measured value is cited—every number is obtained by exhaustive enumeration. No general category theory is developed—only the definition of an adjunction is used, and the adjoint functor theorem is not entered. No point-set topology is developed—Section 4 uses a single instance of a map that fails to extend. The axiom of choice is not discussed—Section 6 separates existence from uniqueness and nothing more. No ranking of value is attached to being an adjoint—it is not claimed that a construction without an adjoint is inferior. Relation to earlier papers: Paper 153 separated three stages of forgetting and exhibited evidence that a right adjoint is absent—this paper counts the paired side, the left adjoints. Paper 201 showed that complete is four different claims—Section 5 separates which of them carries an adjoint. Paper 113 showed that a module is a limit rather than a wall—the neighbouring notion on the same shelf. Paper 149 treated the functoriality of pullback and pushforward—this paper looks at when such a functor has a partner. Paper 179 showed that cannot be constructed has several roots—Section 6 is a fourth case, existing but not unique. Paper 244 puts the same instrument, a count of Hom sets, to a different use—homomorphisms from a cyclic group number six for the dihedral group and two for the quaternion group, separating two groups that share a character table. Here the same instrument measures whether a partner exists—one instrument, two quantities. What is added is the numerical check of an adjunction by counting Hom sets, the naming of one construction that is not even a functor, and the explicit dependence of completion on the class of maps.

1. Notation and placement

Setting: two operations F and U are **adjoint** when, roughly, maps from $F(A)$ to B correspond one to one with maps from A to $U(B)$.

The shape is that an operation F which adds something is paired with an operation U which throws something away. Free groups with forgetting, completion with inclusion, localisation with inclusion.

The adjoint functor theorem is not entered. Only the definition is used. The aim is to name the constructions that are not paired.

2. Check: counting Hom sets confirms the correspondence

Check 2.1: count homomorphisms from the free group F_n on n generators into a group G . If the adjunction holds this is the same as choosing n elements of G freely, so the count must be exactly $|G|^n$.

group G	$ G $	$n = 1$	$n = 2$	$n = 3$
$\mathbb{Z}/6$	6	6	36	216
S_3	6	6	36	216
D_4	8	8	64	512
Q_8	8	8	64	512

Observation 2.2

The count does not depend on the structure of the group at all. $\mathbb{Z}/6$ and S_3 differ in commutativity and both give **216**—seen from a free group, the interior of the target is invisible. Only the number of elements shows.

Observation 2.3

That is what the adjunction means. Counting $\text{Hom}(F_n, G)$ translates completely into choosing n elements of the underlying set—a question on the structured side moved to the set side.

3. Contrast: a non-free source gives no power

Check 3.1: count homomorphisms from $\mathbb{Z}/n$ into G , that is the elements with $g^n = e$.

group G	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\mathbb{Z}/6$	1	2	3	2
S_3	1	4	3	4
D_4	1	6	1	8
Q_8	1	2	1	8

Observation 3.2

The counts are not powers of $|G|$. They are 6 for D_4 and 2 for Q_8 —and now the interior of the target does show. Two groups of the same order 8 returned different numbers.

Observation 3.3

Here is the separator. $\mathbb{Z}/n$ is not free—it imposes the relation $g^n = e$. To the extent that a relation is imposed, the structure of the target enters the answer. Free means carrying no relation at all.

Observation 3.4

The same count becomes an instrument of separation in the neighbouring paper. Paper 244 uses the fact that $\text{Hom}(\mathbb{Z}/2, -)$ returns six for the dihedral group and two for the quaternion group, and separates two groups that share a character table. What this section saw with the same count is that only a free source gives a power of the order—One instrument, a count of Hom , answering can it tell them apart on one side and does it have a partner on the other.

4. The core: one of two compactifications is not even a functor

Setting: embedding a space into a closed one is called **compactification**. There are two.

1. **one-point compactification:** add a **single point** at infinity, turning the line into a circle
2. **Stone--Cech compactification:** add as much as possible

Check 4.1: take the **plain inclusion** of $(0, 1)$ into $\mathbb{R}$ and ask whether it extends to the one-point compactifications.

space	one-point compactification	what the point at infinity meets
$(0, 1)$	circle	**both ends 0 and 1**
$\mathbb{R}$	circle	$\pm\infty$

Observation 4.2

It does not extend. The point at infinity of $(0, 1)$ is approached from both ends. The inclusion sends those two to 0 and 1 in $\mathbb{R}$, and the point at infinity of $\mathbb{R}$ is not there. One point must go to two places.

Observation 4.3

The one-point compactification is therefore not even a functor on continuous maps. It becomes one only when restricted to proper maps—before any question of adjointness, the correspondence fails to carry maps.

Observation 4.4

The Stone–Cech construction, on the other hand, is a left adjoint. Under one word, compactification, stand an adjoint and something that is not even a functor.

5. Completion is not an adjoint until the maps are chosen

Paper 201 showed that complete is four different claims. This paper takes the completion of a metric space and writes down when it is an adjoint.

maps allowed	is completion an adjoint
uniformly continuous	yes
merely continuous	no
Lipschitz	yes

Observation 5.1

One and the same completion stops being an adjoint when the category changes. Writing that completion is a free construction requires saying which category.

Observation 5.2

The reason is simple. A continuous map need not send Cauchy sequences to Cauchy sequences——so there is nothing to extend it to. Uniform continuity is placed there precisely to guarantee that.

6. The algebraic closure is not an adjoint

Every field K has an algebraic closure. It has the same shape as a completion: add what is missing and close up.

Observation 6.1

It is nevertheless not an adjoint. No universal property holds——a map out of $\bar{K}$ into an algebraically closed field is not unique. The algebraic closure of the rationals carries infinitely many automorphisms other than the identity.

Observation 6.2

It exists and is not unique. Paper 179 separated three roots of cannot be constructed——averaging, choice and uncomputability. This is a fourth case——constructible, but with no canonical construction.

Observation 6.3

Failing to be an adjoint is not a defect. The algebraic closure is a perfectly useful construction——what is said here is only that the single phrase free construction erases this difference.

7. Separator and fences

Separator: three stages.

1. does it carry maps at all (is it a functor)——the one-point compactification does not, on continuous maps
2. if it does, does it have a partner (is it an adjoint)——Stone--Cech does
3. if it has one, in which category——completion has one for uniformly continuous maps and not for merely continuous ones

Fences:

1. A construction without an adjoint is not called inferior.
2. The adjoint functor theorem is not entered.
3. The axiom of choice is not discussed——existence is separated from uniqueness, and no more.
4. No point-set topology is developed——one instance of a failure to extend is given.
5. No value is read into the word free.

8. Roll call and closing

1. Homomorphisms out of a free group number exactly $|G|^n$ (Check 2.1): **512** for both D_4 and Q_8 .
2. Structure is invisible in the answer (Observation 2.2): **216** for both $\mathbb{Z}/6$ and S_3 .
3. A non-free source gives no power (Check 3.1): **6** and **2**.
4. The one-point compactification is not even a functor (Observation 4.2): **one point would need two destinations**.
5. Completion stops being an adjoint when the category changes (Section 5).
6. The algebraic closure exists and is not unique (Observation 6.1).

Closing: constructions that add something split into three stages. Those that do not carry maps, those that carry maps without a partner, and those that have one. And under the single word compactification stood a left adjoint and something that is not even a functor. Free construction, completion and closure all look alike and differ in whether a partner exists. Whether one exists can be seen by counting Hom sets——whether the count is $|G|^n$, and nothing more.

Revision Record

Second edition (August 2026): no claim, theorem or check has been altered from the first edition. What is added is a single cross-reference. This paper and Paper 244 (To Determine an Object You Need Every Other One) were published on the same day, so at the time of the first edition neither carried a DOI yet and neither could point to the other. The second edition adds **Observation 3.4** and lists Paper 244, with its DOI, among the references. **One and the same instrument, a count of Hom sets, measures the separation of objects in Paper 244 and the existence of an adjoint here.** The marks used in the body text for caution and emphasis have also been removed, their use not having been consistent across the corpus. **Nothing of substance has changed, and nothing has been withdrawn.**

References

1. Mac Lane, S. (1998). *Categories for the Working Mathematician*, 2nd edition. New York: Springer.
2. Riehl, E. (2016). *Category Theory in Context*. Mineola: Dover.
3. Engelking, R. (1989). *General Topology*, revised edition. Berlin: Heldermann.
4. Bourbaki, N. (1966). *General Topology*, Part 1. Paris: Hermann.
5. Lang, S. (2002). *Algebra*, revised 3rd edition. New York: Springer.
6. Yamagishi, Y. (2026). *Modules Are Not Walls but Limits* [Paper 113]. Zenodo. 10.5281/zenodo.21962996
7. Yamagishi, Y. (2026). *What Fixes the Denominator Is Not the Determinant but the Largest Invariant Factor* [Paper 149]. Zenodo. 10.5281/zenodo.22083590
8. Yamagishi, Y. (2026). *What Was Forgotten Can Be Read Off by Counting Maps* [Paper 153]. Zenodo. 10.5281/zenodo.22084355
9. Yamagishi, Y. (2026). *A Limit Without Its Order Is Not a Quantity* [Paper 154]. Zenodo. 10.5281/zenodo.22084647
10. Yamagishi, Y. (2026). *“Complete” Is Four Separate Statements* [Paper 201]. Zenodo. 10.5281/zenodo.22109052
11. Yamagishi, Y. (2026). *“Cannot Be Constructed” Has Distinct Roots* [Paper 179]. Zenodo. 10.5281/zenodo.22098419
12. Yamagishi, Y. (2026). *To Determine an Object You Need Every Other One* [Paper 244]. Zenodo. 10.5281/zenodo.22159697

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