

“Unsolvable” Has Three Distinct Roots

— Galois, Wantzel, Matiyasevich: the Separator Is Which Tools Were Allowed — [Paper 25]

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ABSTRACT

One word, “unsolvable”, names at least three different things. This paper writes their addresses and shows that the separator is which tools were allowed. NO NEW MATHEMATICAL THEOREM IS CLAIMED. First, the obstruction is a derived series that stalls. The derived series of S_3 and S_4 reach 1 as $6 \rightarrow 3 \rightarrow 1$ and $24 \rightarrow 12 \rightarrow 4 \rightarrow 1$, while that of S_5 stalls at $120 \rightarrow 60$ — on A_5 , a simple non-abelian group of order 60 (computed here) (Section 2). **Second, a brake. Unsolvable, and yet the root is computable.** $x^5 - x - 1$ is irreducible with Galois group S_5 , so it has no expression in radicals. **Its real root is nonetheless 1.16730397826141868425604589985, to as many digits as one asks** (computed here at forty-digit precision) (Section 3). **Third, another brake. “The quintic cannot be solved” is false as stated.** The roots of $x^5 - 2$ are all expressible in radicals, beginning with $2^{1/5}$. **What is true is that there is no general formula in radicals for the general quintic — a statement about the general case and one about a particular case are different claims** (Section 4). **Fourth, change the toolbox and the obstruction changes.** With straightedge and compass a constructible number must have degree a power of 2. Since $2 \cos 20^\circ$ satisfies $t^3 - 3t - 1 = 0$ and so has degree 3, the angle cannot be trisected. **The regular 17-gon, by contrast, can be constructed** ($\varphi(17) = 16 = 2^4$; Gauss 1796) (Section 5). **Fifth, a third toolbox: a finite procedure.** To Hilbert's tenth problem Matiyasevich (1970) answered **that no such algorithm exists.** Here “unsolvable” means neither that a formula is missing nor that a construction is impossible, but that there is no procedure at all (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. Abel–Ruffini, Galois theory, Wantzel's theorem, Gauss's 17-gon and Matiyasevich's theorem are **all standard**. What this paper adds is only **writing addresses for three senses of “unsolvable”, naming the separator as the toolbox, and fencing the widely repeated claim that the quintic cannot be solved, by counterexample.**

1. Notation and position

Write the derived series of a group as $G \supseteq G' \supseteq G'' \supseteq \cdots$ and call G solvable when it reaches the trivial group. A constructible number is one reachable from the rationals by iterated square roots.

Position: Paper 85 traced the line from Galois to Noether. **This paper goes back to Galois's side and shows that what he called impossible has a different root from two other impossibilities.**

2. The obstruction is a derived series that stalls

Verification 2.1

group	orders along the derived series	solvable?
S_3	$6 \rightarrow 3 \rightarrow 1$	yes
S_4	$24 \rightarrow 12 \rightarrow 4 \rightarrow 1$	yes
S_5	stalls at $120 \rightarrow 60$	no
S_6	stalls at $720 \rightarrow 360$	no

(QED)

Observation 2.2

What S_5 stalls on is A_5 , simple and non-abelian of order 60. That is the whole obstruction, and nothing further can be peeled away. Up to the quartic everything peels, so there is a formula; from the quintic it does not, so there is none.

3. A brake: unsolvable, and yet computable

Verification 3.1: $x^5 - x - 1$ is irreducible (checked here) and its Galois group is S_5 . It has one real root, and at forty-digit precision

the real root	1.16730397826141868425604589985
residual $f(\text{root})$	-1.1×10^{-41}

(QED)

Observation 3.2

“Unsolvable” says only that no expression built from the four operations and radicals exists. It does not say the number is out of reach — it can be written to any number of digits.

Observation 3.3

Impossibility is not a property of the object but a relation between the object and the tools (the matter of addresses, Paper 17).

4. Another brake: “the quintic cannot be solved” is false

Verification 4.1

polynomial	irreducible?	solvable in radicals?
$x^5 - 1$	no (cyclotomic)	yes
$x^5 - 2$	yes	yes (one root is $2^{1/5}$)
$x^5 - x - 1$	yes	no (S_5)

(QED)

Observation 4.2

“The quintic cannot be solved” is false as it stands. Solvable quintics are plentiful. The true statement is that there is no general formula in radicals for the general quintic.

Observation 4.3

A claim about the general case must not be traded for a claim about a particular one. This is the same shape as brakes the corpus has applied before — “fails in some case” and “fails in every case” are different.

5. Change the toolbox: straightedge and compass

Verification 5.1: From $2 \cos 3\alpha = (2 \cos \alpha)^3 - 3(2 \cos \alpha)$ with $\alpha = 20^\circ$ and $2 \cos 60^\circ = 1$, the number $t = 2 \cos 20^\circ$ satisfies $t^3 - 3t - 1 = 0$. That cubic is **irreducible over the rationals** (checked here), and numerically

$2 \cos 20^\circ$	1.8793852415718167681
$t^3 - 3t - 1$	9.2×10^{-41}

so $[\mathbb{Q}(2 \cos 20^\circ) : \mathbb{Q}] = 3$. (QED)

Observation 5.2

A constructible number has degree a power of 2 (Wantzel 1837). Three is not a power of two, so the angle cannot be trisected, and doubling the cube fails for the same reason, $2^{1/3}$ having degree three.

Verification 5.3: The regular n -gon is constructible exactly when $\varphi(n)$ is a power of two.

n	$\varphi(n)$	constructible
5	4	yes
7	6	no
9	6	no
17	16	yes (Gauss 1796)
19	18	no
257	256	yes
65537	65536	yes

(QED)

Observation 5.4

With the same tool the heptagon fails and the 17-gon succeeds. What separates them is one number: whether $\varphi(n)$ is a power of two.

6. A third toolbox: a finite procedure

Fact 6.1: Hilbert's tenth problem asks whether there is an algorithm deciding if a given Diophantine equation has an integer solution. Matiyasevich (1970) answered **no**.

Observation 6.2

Here “unsolvable” differs again from both earlier senses. No formula is missing and no construction is impossible: there is no procedure at all. More digits will not help, and neither will more patience — quite unlike the quintic of Section 3, which came out to thirty digits.

7. The census: three roots of one word

Observation 7.1

sense of “unsolvable”	tools allowed	the obstruction	theorem
not solvable in radicals	the four operations and radicals	the Galois group is not solvable	Abel–Ruffini, Galois
not constructible	straightedge and compass	the degree is not a power of 2	Wantzel 1837
undecidable	a finite procedure	there is no procedure	Matiyasevich 1970

Observation 7.2

The separator is not how hard the problem is. It is which tools were allowed. Change the toolbox and the same object moves from impossible to routine — the quintic's root lies beyond radicals and thirty digits from a computer.

8. Conclusion

1. The obstruction is a stalling derived series (Ver. 2.1): S_5 stalls on A_5 .
2. Unsolvable, and yet computable (Ver. 3.1): 1.16730397826141868425604589985.
3. “The quintic cannot be solved” is false (Ver. 4.1): $x^5 - 2$ can.
4. With straightedge and compass the fence is a power of two (Ver. 5.1, 5.3): no heptagon, but a 17-gon.
5. Undecidable means there is no procedure at all (Fact 6.1).
6. The separator is which tools were allowed (Obs. 7.1).

Closing: “Impossible” is not a label stuck on an object. It is the name of a relation between the object and the tools one brought. To say “unsolvable” without naming the tools is to have said nothing yet.

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