

Special in Two Dimensions Is Not One Phenomenon

— Inversion Preserves Harmonicity Without a Factor Only at $d=2$: One and the Same d Minus Two Appears in Three Places, and Huygens Is Not Among Them — [Paper 251]

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— *From thinking to syncing.* —

Abstract

Special in two dimensions appears again and again in this body of work: the absence of a steady absorber in Paper 249, the Huygens principle in Paper 247, and the conformal invariance of harmonic functions. This paper asks whether they share a root: and finds that two of them do, while one was not about $d = 2$ at all. No new mathematical theorem and no new law is claimed. First, the mean value property holds exactly. In three dimensions the spherical mean of $1/|x - p|$ differs from the value at the centre by at most 4×10^{-16} , for radii from 0.5 to 1.9 (Section 3). Second, the failure follows a formula. For a function that is not harmonic the discrepancy is exactly $R^2 \Delta u / (2d)$: **2.890000000000** at $d = 5$, matching the formula (Section 3). Third, this is the core. Inversion alone preserves harmonicity only at $d = 2$. Pulling back by $x \mapsto x/|x|^2$ with no factor gives a Laplacian of -4.2×10^{-7} at $d = 2$, that is zero, and -1.728 at $d = 3$: multiplying by r^{2-d} restores -5.6×10^{-7} at $d = 3$ (Section 4). Fourth, one and the same $d - 2$ appears in three places. The exponent $2 - d$ of the fundamental solution, the factor r^{2-d} of the Kelvin transform, and the conformal weight $(d - 2)/2$ of the Laplacian: all three vanish at $d = 2$ (Section 5). Fifth, Huygens does not belong here. A wave leaves a tail at $d = 1$ and in every even dimension, so it is not specific to $d = 2$: a statement about parity that happens to include $d = 2$ (Section 6). Sixth, the exponent $n = d - 1$ of Paper 1 does not degenerate at $d = 2$. It is simply 1, and what vanishes is the exponent of the potential, one derivative away (Section 7).

Scope of this paper (scope note) : No new mathematical theorem and no new law is claimed: the mean value property of harmonic functions, the maximum principle, the Kelvin transform, the restriction of conformal maps to Mobius transformations for $d \geq 3$ (Liouville, 1850), and the validity of the Huygens principle in odd space dimensions are all standard. No measured value is cited: every number is computed from a definition. No conformal geometry is developed: Section 5 touches only the fact that one $d - 2$ appears in three places and does not enter the theory of conformal invariants. It is not claimed that $d = 2$ is essentially special. What is shown is only which mechanism degenerates at $d = 2$ and which does not. No potential theory is built: Sections 2 and 3 confirm known equivalences with numbers. No complex analysis is used: the check in Section 4 measures a Laplacian by finite differences and does not rest on the theory of holomorphic functions. Relation to earlier papers: Paper 249 showed that no steady absorber exists in two dimensions. This paper sees that the same $d - 2$ underlies it, the Kelvin factor and the conformal weight. Paper 156 traced the flatness inside a shell to harmonicity and wrote that it is not the mean value theorem that does it. This paper measures that harmonicity itself as a function of dimension. Paper 247 showed that splitting by parity is not one phenomenon: Section 6 confirms that Huygens belongs to that family and not to this one. Paper 1 established $n = d - 1$: Section 7 confirms that it does not degenerate at $d = 2$. Paper 111 showed that light determines geometry only up to conformal structure: the conformal weight of Section 5 stands beside it. What is added is a numerical demonstration that inversion preserves harmonicity without a factor only at $d = 2$, a single table showing three degeneracies coming from one $d - 2$, and the removal of Huygens and of $n = d - 1$ from that family.

1. Notation and placement

Setting: a function whose Laplacian vanishes is called **harmonic**. **Electrostatics, steady diffusion and the interior of a spherical shell are three faces of that one property** (Papers 156 and 249).

Question: special in two dimensions appears again and again here. **The absence of a steady absorber in Paper 249, the Huygens principle in Paper 247, and the conformal invariance of harmonic functions. Do they share a root?**

Two of them do, and one was not about $d = 2$ at all. This paper is that adjudication.

It is not claimed that $d = 2$ is essentially special. Only which mechanism degenerates there, and which does not.

2. Three ways of saying harmonic

Being harmonic is equivalent to each of the following.

1. The Laplacian vanishes.
2. The mean over any sphere equals the value at the centre (the mean value property).
3. There is no interior maximum and no interior minimum (the maximum principle).

Observation 2.1

The second is an equivalence and not a one-way implication. A continuous function with the mean value property is necessarily harmonic, so either direction may serve as the definition.

Observation 2.2

Paper 156 wrote that the flatness inside a shell is not due to the mean value theorem. That was about what forces the constant. This paper measures how that harmonicity itself behaves with dimension. A different axis, and no collision.

3. Check: the mean value property is exact, and so is its failure

Check 3.1: use harmonic functions that are not polynomials, $u(\mathbf{x}) = 1/|\mathbf{x} - \mathbf{p}|$ in three dimensions and $u(\mathbf{x}) = \log |\mathbf{x} - \mathbf{p}|$ in two, with deterministic quadrature.

dimension	R	spherical mean	value at centre	difference
3	0.5	0.36612601296911	0.36612601296911	0
3	1.0	0.36612601296911	0.36612601296911	1.7×10^{-16}
3	1.9	0.36612601296911	0.36612601296911	3.9×10^{-16}
2	1.9	0.97080761238622	0.97080761238622	1.1×10^{-16}

Check 3.2: for a function that is not harmonic the discrepancy should be $R^2 \Delta u / (2d)$. Take $R = 1.7$.

function	value at centre	spherical mean	difference	formula
$d = 2: x^2 - y^2$ (harmonic)	0.050000000000	0.050000000000	0	0
$d = 3: x^2 + y^2 - 2z^2$ (harmonic)	-0.370000000000	-0.370000000000	0	0
$d = 4: x_1^2 + x_2^2 - x_3^2 - x_4^2$ (harmonic)	-0.130000000000	-0.130000000000	0	0
$d = 3: x^2$ (not harmonic)	0.090000000000	1.053333333333	0.963333333333	0.963333333333
$d = 5: x ^2$ (not harmonic)	0.550000000000	3.440000000000	2.890000000000	2.890000000000

Observation 3.3

Zero when harmonic, and exactly the formula when not.The mean value property is not a yes-or-no matter; the size of its failure is fixed too.

Observation 3.4

The maximum principle was checked as well.Solving Laplace’s equation on a square, the boundary maximum is 1.476028904 and the interior maximum is 1.438591762: the interior exceeded the boundary by 0.

4. The core: inversion alone preserves harmonicity only at $d = 2$

Setting: pull a function back by the inversion $x \mapsto x/|x|^2$. Is the pull-back of a harmonic function harmonic again?

Check 4.1: $u(x) = x_1$ is harmonic in every dimension. Measure the Laplacian of the pull-back with no factor and multiplied by r^{2-d} (the Kelvin transform).

d	no factor	multiplied by r^{2-d}
2	-4.219×10^{-7}	-4.219×10^{-7}
3	-1.728	-5.551×10^{-7}
4	-3.169	-7.550×10^{-7}
5	-3.959	-8.549×10^{-7}
6	-2.898	-3.109×10^{-7}

Observation 4.2

Only at $d = 2$ is it zero without the factor. From $d = 3$ upward harmonicity breaks unless the factor is supplied: -1.728 is not an error; the function really is not harmonic.

Observation 4.3

The reason is plain. The factor is r^{2-d} , and at $d = 2$ that is $r^0 = 1$. Multiplying changes nothing, so there is nothing to multiply by.

Observation 4.4

This is the content of conformal invariance only at $d = 2$. In two dimensions a holomorphic pull-back preserves harmonicity: pulling $u = x^2 - y^2$ back by $f(z) = z^2$ gives Laplacians of 1.5×10^{-7} , 3.1×10^{-7} and 1.7×10^{-7} at three points.

5. One and the same $d - 2$ in three places

The exponent of the factor r^{2-d} from Section 4 turns up elsewhere too.

where it appears	value at $d = 2$	what happens then
the exponent $2 - d$ of the fundamental solution	0	the power disappears and a logarithm takes its place
the factor r^{2-d} of the Kelvin transform	1	nothing needs multiplying
the conformal weight $(d - 2)/2$ of the Laplacian	0	conformal invariance as it stands

Check 5.1: watch the fundamental solution degenerate. $(r^{2-d} - 1)/(2 - d)$ tends to $\log r$ as $d \rightarrow 2$.

$d - 2$	value at $r = 2$	difference from $\log 2$
10^{-1}	0.669670084632	2.35×10^{-2}
10^{-2}	0.690750456296	2.40×10^{-3}
10^{-4}	0.693123158464	2.40×10^{-5}
10^{-6}	0.693146940289	2.40×10^{-7}
10^{-8}	0.693147172992	7.57×10^{-9}

Here $\log 2 = 0.693147180560$.

Observation 5.2

All three come from the same $d - 2$ vanishing. They are therefore three faces of one root, not three roots. The absence of a steady absorber in Paper 249 also follows from this degeneration of the fundamental solution.

Observation 5.3

The size of the conformal group enters here. For $d \geq 3$ conformal maps are Mobius transformations, and that group has dimension $(d + 1)(d + 2)/2$: 10 at $d = 3$, 15 at $d = 4$, 66 at $d = 10$. The same formula returns 6 at $d = 2$, while the truth is infinite dimensional, every holomorphic map being conformal. Formula and truth part company exactly at $d = 2$.

6. Huygens does not belong here

Paper 247 counted the dimensions in which a wave leaves a tail. It does at $d = 2$, so it looks like the same family.

d	does the wave leave a tail
1	yes
2	yes
3	**no**
4	yes
5	**no**
6	yes
7	**no**
8	yes

Observation 6.1

Tails occur at $d = 1$ and in every even dimension. This is not specific to $d = 2$: a statement about parity that happens to include $d = 2$.

Observation 6.2

What separates them is the support of the Green function, a different mechanism from the $d - 2$ of this paper. The factor $d - 2$ does not vanish at $d = 4$ or $d = 6$, while Huygens fails at both: the two hit different sets of dimensions.

Observation 6.3

Lumping them together therefore reads a statement about all even dimensions as a statement about two.

7. The exponent $n = d - 1$ does not degenerate at $d = 2$

One more looks like family and is not. The $n = d - 1$ of Paper 1.

d	force exponent $n = d - 1$	potential exponent $2 - d$
2	1	0
3	2	-1
4	3	-2
5	4	-3
6	5	-4

Observation 7.1

At $d = 2$ the exponent is 1, an ordinary value, and nothing degenerates. What vanishes is the exponent of the potential, one derivative away from the force.

Observation 7.2

Paper 249 stopped in two dimensions because the potential became a logarithm, not because the force exponent became 1. One derivative separates degenerating from not.

8. Separator and fences

Separator: whether $d - 2$ vanishes, or whether the parity of d is at work. The two are told apart by the set of dimensions they hit: $d - 2$ vanishes only at $d = 2$, while parity hits $d = 2, 4, 6, \dots$ alike.

Fences:

1. It is not claimed that $d = 2$ is essentially special: only which mechanism degenerates.
2. The theory of conformal invariants is not entered.
3. No complex analysis is used: a Laplacian was measured by finite differences.
4. No potential theory is built: known equivalences were confirmed with numbers.

5. It is not claimed that every instance of special in two dimensions was counted: five were examined.

9. Roll call and closing

1. The mean value property is exact (Check 3.1): differences at most 4×10^{-16} .
2. Its failure follows the formula (Check 3.2): **2.890000000000** at $d = 5$.
3. Inversion passes without a factor only at $d = 2$ (Check 4.1): -4.2×10^{-7} against -1.728 .
4. One $d - 2$ in three places (Section 5): **exponent, factor, conformal weight**.
5. The conformal group is infinite dimensional only at $d = 2$ (Observation 5.3): **the formula returns 6, the truth is infinite**.
6. Huygens fails in every even dimension (Section 6): **not specific to $d = 2$** .
7. $n = d - 1$ does not degenerate at $d = 2$ (Section 7): **it is simply 1**.

Closing: special in two dimensions was not one phenomenon. And once counted, the breakdown differed from what was expected. Of the three that seemed to be there, two are faces of one $d - 2$, and the remaining one, Huygens, was not about $d = 2$ at all: a statement about every even dimension that happened to include two. And $n = d - 1$, which looked like family, degenerates not at all. There is one way to tell them apart: write down the set of dimensions each one hits. Only $d = 2$, or every even dimension, or none at all. Write the set, and it becomes visible that different things were living under one phrase.

Revision Record

Second edition (September 2026): no claim, theorem or check has been altered from the first edition. In the English references, the lines citing earlier papers of this corpus still carried their Japanese titles; **they now carry the English titles.** **Nothing of substance has changed, and nothing has been withdrawn.**

The dashes used as separators in the text were replaced by punctuation (October 2026). No claim, number or reference was changed.

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