

That Knots Come Undone in Four Dimensions Is Not One Phenomenon

— Linking Asks $p + q = n - 1$ and Knotting Asks $n - k = 2$: For Circles in Three Dimensions the Two
Happen to Coincide — [Paper 254]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

In four dimensions every knot in a string comes undone. And in four dimensions a surface can be knotted. This paper asks whether the two share a root—the answer is no: two separate conditions merely coincide for circles in three dimensions. No new mathematical theorem and no new law is claimed. First, the instruments were built here and checked against known values. Integrating the Gauss linking number with 4000 steps gives -1.000000 for the Hopf link and 0.000000 for two separated circles (Section 2). Second, this is the core. Count how often one loop pierces a disk spanned by the other. In three dimensions, **1**. Placed in four and pushed by 10^{-9} along the fourth coordinate, **0** (Section 3). Third, the difference is robust. Under random pushes the count was **1** in all 200 trials in three dimensions and **0** in all 200 in four (Section 4). Fourth, linking is a matter of addition, not of dimension. An S^1 pierces a ball spanned by an S^2 **1** time in four dimensions and **0** times in five—the condition is $p + q = n - 1$ (Section 5). Fifth, and yet surfaces knot. Evaluating the Alexander polynomial of the trefoil at integers gives $t^2 - t + 1$, which is not identically **1**—and the spun trefoil, an S^2 in four dimensions, carries the same polynomial (Section 6). Sixth, the two conditions are different. Linking asks $p + q = n - 1$ and knotting asks $n - k = 2$ —for circles in three dimensions both return **2**, and that is why they looked like one (Sections 7 and 8).

Scope of this paper (scope note) : No new mathematical theorem and no new law is claimed—the Gauss linking number, the unknotting of spheres in codimension three or more (Zeeman), Artin's spinning, and the Alexander polynomial are all standard. No knot theory is built—what is used is one integral, a count of intersections, and a 2×2 determinant. Higher dimensional knots are not classified. It is not claimed that the Alexander polynomial is a complete invariant—distinct knots share polynomials. Section 6 uses only the one direction, that a knot with $\Delta \neq 1$ cannot be undone. The spun trefoil is not constructed here—Artin's construction and the fact that it carries the same polynomial are cited. The count of intersections is not the linking number—Section 3 counts unsigned crossings; the signed invariant is the linking number of Section 2. The Hopf fibration is not treated—the Hopf link of this paper shares only its name and has nothing to do with the three Hopfs of Paper 125. The numbers are discretised—4000 steps for the linking number and 20000 for the crossing count. The relation to earlier papers. Paper 251 showed that special in two dimensions is not one phenomenon—this paper stands beside it and separates special in four. Paper 247 counted the splittings by parity as eight—the same shape of roll call. Papers 69, 105 and 114 treated the specialness of four, and 105 offered exotic $\mathbb{R}^4$ —the unknotting here is a distinct root: it uses no deep exception of smooth structure and follows from counting codimensions alone. Papers 173 and 185 wrote where codimension does the work—this paper adds one to that family. What is added is building the linking integral and the crossing count and checking them against known values, seeing the crossing vanish at the concrete size 10^{-9} , measuring the S^1 and S^2 link in four and in five dimensions, and setting linking and knotting side by side as two conditions that coincide for circles in three dimensions.

1. Notation and standing

The setting: place a k -sphere S^k inside a space of n dimensions. The number $n - k$ is its codimension. A string is S^1 and a surface is S^2 .

The question: in four dimensions every knot in a string comes undone. And in four dimensions a surface can be knotted. Do the two share a root?

They do not. Linking and knotting are different conditions, and for circles in three dimensions both happen to hold.

No knot theory is built here. What is used is one integral, a count of intersections, and a 2×2 determinant.

2. Building the instruments and checking them

Check 2.1: integrate the Gauss linking number with 4000 steps and compare with the exact values.

configuration	linking number	exact
the Hopf link	−1.000000	−1
two separated circles	0.000000	0

Observation 2.2

Agreement to six decimals. The instrument used below is calibrated.

Observation 2.3

The linking number is a signed quantity. Section 3 counts unsigned crossings, a different thing—the two are not to be conflated.

3. The core: the crossing vanishes under a push of 10^{-9}

The setting: one loop C_1 spans a disk. **Count how often the other loop pierces it. Piercing means it cannot be pulled free, not piercing means it can**—that is what linking naively says.

Check 3.1: counted with **20000** steps.

configuration	crossings
the Hopf link in three dimensions	1
two separated circles in three dimensions	0
the same link placed in four dimensions ($x_4 = 0$)	1

Check 3.2: keeping it in four dimensions, push it by ε along the fourth coordinate.

ϵ	crossings	linking number of the three-dimensional shadow
0	1	−1.000000
10^{-9}	0	−1.000000
10^{-6}	0	−1.000000
10^{-3}	0	−1.000000
10^{-1}	0	−1.000000

Observation 3.3

It vanishes at 10^{-9} . In four dimensions no push is too small: no crossing survives.

Observation 3.4

Meanwhile the linking number of the shadow stays at −1.000000. The shadow goes on looking linked while the thing itself has come apart——what has come apart in four dimensions cannot be drawn as apart in three.

Observation 3.5

The reason is a count. A disk has 2 dimensions and a loop has 1. In general position inside n dimensions they meet in $2 + 1 - n$ dimensions——at $n = 3$ that is 0, so points survive; at $n = 4$ it is −1, that is, nothing.

4. The difference is robust

Check 4.1: push at random, 200 times. In four dimensions a normal deviate of standard deviation 0.05 is added to the fourth coordinate.

space	trials	crossings
three dimensions	200	always 1
four dimensions	200	always 0

Observation 4.2

In three dimensions it cannot be removed; in four almost any push removes it. This is a property of the dimension, not of one arrangement.

5. Linking is a matter of addition, not of dimension

The setting: replace the partner by a surface. The unit S^2 inside $x_4 = 0$ spans a three-dimensional ball. **Count how often a loop pierces that ball.**

Check 5.1: build a loop in four dimensions that pierces the ball exactly once.

configuration	piercings
S^1 and S^2 in four dimensions	1
moved into five dimensions and shifted by 10^{-9}	0

Observation 5.2

Two circles do not link in four dimensions, and yet a loop and a surface do. Nothing links in four dimensions is false.

Observation 5.3

The condition is written with addition, not with a dimension. S^p and S^q link when $p + q = n - 1$.

the pair	n	$p + q$ against $n - 1$	do they link
S^1 and S^1	3	2 against 2	yes
S^1 and S^1	4	2 against 3	no
S^1 and S^2	4	3 against 3	yes
S^2 and S^2	4	4 against 3	no
S^1 and S^2	5	3 against 4	no

6. And yet surfaces knot

The setting: linking concerns two objects. **Knotting concerns one**—whether a single loop is tied in a way that cannot be undone.

Check 6.1: from the Seifert matrix V of the trefoil—the 2×2 matrix with top row $-1, 1$ and bottom row $0, -1$ —evaluate $\Delta(t) = \det(V - tV^T)$ at integers.

t	$\det(V - tV^T)$	$t^2 - t + 1$
0	1	1
1	1	1
2	3	3
3	7	7
-1	3	3
5	21	21

Observation 6.2

$\Delta(t) = t^2 - t + 1$ is not identically 1. The unknot has $\Delta = 1$, so the trefoil cannot be undone.

Observation 6.3

Spinning the trefoil in Artin's sense makes an S^2 in four dimensions carrying the same polynomial. In four dimensions strings always come undone, and yet surfaces knot.

Observation 6.4

The Alexander polynomial is not a complete invariant——distinct knots share polynomials. Only the one direction is used here: $\Delta \neq 1$ means it cannot be undone.

7. The two conditions are different formulas

The knotting condition is $n - k = 2$ (in codimension three or more everything comes undone, Zeeman).

object	n	$n - k$	can it be knotted
S^1 in three dimensions	3	2	yes
S^1 in four dimensions	4	3	no, always undone
S^2 in four dimensions	4	2	yes
S^2 in five dimensions	5	3	no, always undone
S^3 in five dimensions	5	2	yes
S^3 in six dimensions	6	3	no, always undone

Observation 7.1

Set the two side by side. Linking asks $\mathbf{p} + \mathbf{q} = \mathbf{n} - 1$ and knotting asks $\mathbf{n} - \mathbf{k} = 2$. One is a sum of two objects' dimensions and the other is one object's codimension: they are not counting the same thing.

Observation 7.2

Independence follows from a case where only one holds. In four dimensions an S^1 and an S^2 do link, while that same S^1 cannot be knotted——linking holds and knotting does not.

8. For circles in three dimensions the two coincide

the question	the condition at work	the answer
do two circles link in three dimensions	$p + q = n - 1: 2 = 2$	yes
can a circle be knotted in three dimensions	$n - k = 2: 3 - 1 = 2$	yes
do two circles link in four dimensions	$p + q = n - 1: 2 \neq 3$	no
can a circle be knotted in four dimensions	$n - k = 2: 4 - 1 = 3$	no
do a loop and a surface link in four dimensions	$p + q = n - 1: 3 = 3$	yes
can a surface be knotted in four dimensions	$n - k = 2: 4 - 2 = 2$	yes

Observation 8.1

For circles in three dimensions both formulas return exactly 2. That is why they looked like one story.

Observation 8.2

Going up to four they part. For circles both are lost; for surfaces only knotting returns——the sentence knots come undone in four dimensions says two different things at once.

Observation 8.3

Paper 105 offered exotic $\mathbb{R}^4$ as the exception of four dimensions. The result here is a distinct root——it uses no deep exception of smooth structure and follows from two subtractions, $2 + 1 - n$ and $n - k$. Under the one phrase special in four dimensions there live at least three different things.

9. Roll call and closing

1. **The instrument is calibrated** (Check 2.1): -1.000000 and 0.000000 .
2. **The crossing vanishes at 10^{-9}** (Check 3.2): **1 in three dimensions, 0 in four.**
3. **The shadow does not move** (Observation 3.4): **the linking number stays -1.000000 .**
4. **Robust under perturbation** (Check 4.1): **1 and 0 in all 200 trials.**
5. **A loop and a surface link in four** (Check 5.1): **once, and not at all in five.**
6. **Surfaces knot** (Check 6.1): $\Delta(t) = t^2 - t + 1 \neq 1$.
7. **The two conditions differ** (Observation 7.1): $p + q = n - 1$ against $n - k = 2$.
8. **They coincide for circles in three** (Observation 8.1): **both return 2.**

Closing: that knots come undone in four dimensions was not one phenomenon. Whether things link is settled by the sum of two objects' dimensions; whether a thing can be knotted is settled by one object's codimension. Separate subtractions, and neither implies the other. For circles in three dimensions both return 2, so they looked like a single fact. One thing separates them——write down what is being counted. A sum of two dimensions, or one codimension? Write it down, and that strings come undone while surfaces knot in the same four dimensions is seen to be neither a contradiction nor a coincidence.

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