

Stationary Is Not Equilibrium, and Detailed Balance Is Only Sufficient

— A Distribution Can Stop Moving While the Probability Keeps Going Round — What Has Settled Is the Distribution, Not the Flow — [Paper 263]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

Abstract

When a distribution has stopped moving, the system is said to have settled. This paper asks what has settled—the answer is the distribution alone. The probability can keep going round a ring while not one entry of the distribution moves. Detailed balance is sufficient for stationarity and not necessary. No new mathematical theorem and no new law is claimed. First, build a ring of three states. Take 0.5 clockwise, 0.2 anticlockwise and 0.3 to stay—the uniform distribution then has inflow **0.233333** and outflow **0.233333**, and is stationary (Section 2). Second, this is the core of the paper. Detailed balance asks for $\pi_i P_{ij} = \pi_j P_{ji}$, and here that reads **0.166667** against **0.066667**, which do not agree—stationary, and detailed balance broken (Section 3). Third, a circulation remains.

$J = (p - q)/3 = 0.100000$ flows on at every step—the distribution does not move and the probability is going round (Section 3). Fourth, running it forward does not remove it. Starting uniform and advancing 1000 steps leaves **0.3333333333**, and the circulation stays at 0.100000 (Section 4). Fifth, a lopsided start lands in the same place. It settles to uniform in 20 steps, and the circulation is still 0.100000 there—what relaxed was the distribution, not the flow (Section 4). Sixth, so detailed balance is not a requirement. Setting $p = q$ restores it, but that is merely imposing a stronger condition—stationarity was not asking for it (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the definition of a stationary distribution, that detailed balance is sufficient for stationarity, and the existence of non-reversible Markov chains are all standard. No probability theory is built—what is used is a 3×3 matrix and repeated multiplication. Rates of convergence are not discussed—neither relaxation times nor spectral gaps are treated. What is examined is what remains after settling, and no more. The design of non-reversible MCMC is not treated—that dropping detailed balance can be faster is mentioned, but how to design such a chain is not. Thermodynamic irreversibility is not conflated with this—the irreversibility of Papers 04, 131 and 248 is on the thermodynamic side, about erasure and work. The reversibility here is the symmetry of microscopic transition probabilities, a different level. Fluctuation-dissipation and Onsager are not entered—Papers 78, 89, 106 and 136 treat fluctuation-dissipation and Papers 120 and 143 treat Onsager. Those concern reciprocity near equilibrium, whereas this paper concerns the condition of microscopic reversibility itself. Ergodicity is not treated—it appears 9 times in earlier papers, but this one merely places an irreducible aperiodic finite chain and discusses no property of it. No physical system is modelled—the ring of three states is a constructed example, not a particular physical system. Relation to earlier papers: Paper 143 showed that minimum entropy production is not a principle but a theorem with a fence—this paper likewise asks after the standing of something called a principle, here detailed balance. Paper 201 counted “complete” as four different claims—this paper separates “has settled” into two different claims. Paper 248 asked whose cost the demon's is—this paper asks after no cost, and looks only at the flow that remains. Paper 190 measured rare on a logarithmic scale—this paper conversely reports the digits that do not move. What is added is constructing an explicit chain that is stationary while detailed balance fails, putting the circulation at the number 0.100000, running it forward to confirm that the circulation survives the relaxation of the distribution, and writing that restoring detailed balance means imposing a stronger condition.

1. Notation and Placement

Setting: consider a Markov chain on finitely many states. A distribution π is stationary when advancing one step leaves it unchanged. Detailed balance holds when $\pi_i P_{ij} = \pi_j P_{ji}$ for every pair of states.

Question: are these the same thing?

The answer is no. Detailed balance implies stationarity—but stationarity does not imply detailed balance. It is sufficient and not necessary.

A caution: reversibility in this paper means the symmetry of transition probabilities. It is a different level from the thermodynamic irreversibility of Papers 04, 131 and 248, the same word applied elsewhere.

2. Building a Ring of Three States

Setting: arrange states **1, 2, 3** in a ring, with $p = 0.5$ clockwise, $q = 0.2$ anticlockwise and **0.3** to stay. The transition matrix is

$$P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

Every row sums to 1, and so does every column.

Check 2.1: for the uniform distribution $\pi = (1/3, 1/3, 1/3)$, count the inflow to and outflow from one state.

Quantity	Value
Inflow $\pi p + \pi q$	0.233333
Outflow $\pi(p + q)$	0.233333
Difference	2.8×10^{-17} (rounding)

Observation 2.2

the uniform distribution is stationary. This follows from the columns summing to 1 (a doubly stochastic matrix) and does not depend on the values of p and q .

3. The Core: Stationary While Detailed Balance Fails

Check 3.1: test $\pi_i P_{ij} = \pi_j P_{ji}$ on a neighbouring pair.

Edge	Value
$\pi_1 P_{12} = (1/3)(0.5)$	0.166667
$\pi_2 P_{21} = (1/3)(0.2)$	0.066667
Do they agree?	no

Observation 3.2

stationarity was confirmed in Section 2, and detailed balance fails here. The sentence sufficient but not necessary is displayed by these two tables.

Check 3.3: count the net flow across an edge.

Quantity	Value
Clockwise flow πp	0.166667
Anticlockwise flow πq	0.066667
Net circulation $J = (p - q)/3$	0.100000

Observation 3.4

a probability of 0.100000 goes round the ring one way at every step. And the contents of each state stay at $1/3$ —because as much comes in as goes out.

Observation 3.5

the separator—what has to be changed for these to become different things. Whether the flow of probability closes a loop. If it closes, a circulation remains; if not, each edge balances on its own.

4. Running It Forward Does Not Remove the Circulation

Check 4.1: start from the uniform distribution and advance.

Steps	Distribution
1	(0.3333333333, 0.3333333333, 0.3333333333)
10	(0.3333333333, 0.3333333333, 0.3333333333)
1000	(0.3333333333, 0.3333333333, 0.3333333333)

Check 4.2: start from the lopsided $(1, 0, 0)$ and advance.

Steps	Distribution	Circulation J
1	(0.300000, 0.500000, 0.200000)	transient
5	(0.332630, 0.334120, 0.333250)	transient
20	(0.333333, 0.333333, 0.333333)	0.100000
100	(0.333333, 0.333333, 0.333333)	0.100000

Observation 4.3

in 20 steps the distribution settles to uniform. The circulation stays at 0.100000 and does not fall at all—what relaxed was the distribution, not the flow.

Observation 4.4

the single phrase “it has settled” names two different things—that the distribution has stopped moving and that the flow has stopped. Here only the first has happened.

5. Detailed Balance Is Not a Requirement

Check 5.1: move q and set the two conditions side by side.

q	Stationary?	Detailed balance?	Circulation J
0.0	yes	no	0.166667
0.2	yes	no	0.100000
0.4	yes	no	0.033333
0.5	yes	yes	0.000000

Observation 5.2

every row is stationary. Detailed balance holds in the single row where $p = q = 0.5$ —the set of stationary chains is the wider one, and detailed balance is one point inside it.

Observation 5.3

setting $p = q$ restores detailed balance. But that is merely imposing a stronger condition, and stationarity was not asking for it—the same figure Paper 143 wrote for minimum entropy production, where what is called a principle turns out to be a chosen condition.

Observation 5.4

what this implies in practice—MCMC uses detailed balance because it is easy to design with, not because correctness requires it. A non-reversible design that drops it can be faster—this paper does not treat how to design one, but the table above shows that there is room to drop it.

6. Roll Call and Closing

1. The uniform distribution is stationary (Check 2.1): inflow and outflow both 0.233333.
2. It does not depend on p and q (Observation 2.2): the matrix is doubly stochastic.
3. Detailed balance fails (Check 3.1): 0.166667 against 0.066667.
4. A circulation remains (Check 3.3): $J = (p - q)/3 = 0.100000$.
5. Unmoved after 1000 steps (Check 4.1): still 0.3333333333.
6. A lopsided start clears in 20 (Check 4.2): but the circulation does not.
7. Only the distribution relaxed (Observation 4.3): the flow is still 0.100000.
8. The stationary set is wider (Check 5.1): four rows stationary, one with detailed balance.
9. The separator is whether the loop closes (Observation 3.5): if it closes, a circulation remains.

Closing: the single phrase “it has settled” named two different things—that the distribution has stopped moving and that the flow has stopped. On the ring of three states the distribution stands perfectly still at $1/3$, while a probability of 0.100000 goes round at every step. Start from a lopsided distribution and it settles to uniform in 20 steps, and the circulation there has not fallen at all. Detailed balance is sufficient for stationarity and not necessary, and setting

$p = q$ to recover it is imposing on oneself a stronger condition that stationarity never asked for. One thing separates them—writing down whether it is the distribution or the flow that has stopped. Write it down, and the occasions for imposing detailed balance separate from those where it may be dropped. Do not write it down, and one calls a system in which 0.100000 is still going round “at equilibrium”.

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Yuuki YAMAGISHI (山岸 勇輝)

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