

What Separated the Two Low-Temperature Models Is Not Quantisation but the Density of States

— Both Quantise Exactly as Much — One Line, $g(\omega)$, Differs, and the **3** in T^3 Is the Dimension of Space —

[Paper 265]

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— *From thinking to syncing.* —

Abstract

Einstein (1907) and Debye (1912) give entirely different answers for the specific heat at low temperature. This paper asks what separated the two—the answer is not quantisation. Both quantise the oscillators exactly as much, and the only difference is the density of states $g(\omega)$. No new mathematical theorem and no new law is claimed. First, count how far apart they are at low temperature. At $T/\theta = 0.05$ Debye gives 9.740757×10^{-3} and Einstein 8.244615×10^{-7} —a factor of **11814.7** (Section 2). Second, at high temperature they cannot be told apart. At $T/\theta = 1.00$ they give 0.951732 and 0.920674, a ratio of **1.0**—they part company only at low temperature (Section 2). Third, this is the core of the paper. The form of the departure differs. Debye is a power and Einstein an exponential—apply the same measurement to Einstein and a meaningless “exponent” of **34.067** comes out (Section 3). Fourth, the **3** in T^3 is the dimension. Solving in d dimensions and measuring the low-temperature exponent gives **1.000137**, **2.000000**, **3.000000** and **3.999998**—the exponent is the dimension itself (Section 4). Fifth, that dimension comes from the density of states. Since $g(\omega) \propto \omega^{d-1}$, it follows that $C \propto T^d$ —and d never once appears in the quantisation (Section 4). Sixth, a 4π sits inside that density of states. From the sphere in k -space comes $g(\omega) \propto 4\pi k^2$ —and $4\pi = \mathbf{12.566371}$ is the whole solid angle of the sphere (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the Einstein model, the Debye model, the low-temperature T^3 law, and the derivation of the density of states are all standard.No solid-state physics is built—what is used is two integrals and the measurement of an exponent by a log ratio.Lattice dynamics is not treated—the dispersion is simply taken as linear, and neither real phonon dispersion, nor optical branches, nor anharmonic terms are treated.Neither model is called correct—that Debye is closer to measurement is cited, but both are approximations and their relative merit is not the subject here.The electronic specific heat is not treated—metals carry a contribution linear in T from the electrons, and this paper looks only at the lattice contribution.No value of the Debye temperature is claimed—everything is written in the ratio T/θ , with no θ of any particular substance.The numbers are discretised—the integrals use the trapezoidal rule with 40000 steps, and the comparison uses the two points $T/\theta = 0.02$ and 0.04 .The Debye length is not conflated with this—the 10 occurrences of Debye in earlier papers are the Debye length of a plasma (Papers 22, 52 and 54), a different work by the same man from the lattice model here.Bloch and Brillouin are likewise not conflated—the 19 occurrences of Bloch are the Bloch sphere of a qubit and Brillouin appears as wave propagation and the price of observation; neither concerns crystals.Relation to earlier papers: Paper 02 showed that a pure solid angle 4π appears in an inverse-square field—Section 5 here writes that the same 4π also sits inside the density of states.Paper 258 wrote the condition for 4π to appear as three items—this section is a place where those three hold.Papers 117 and 120 treated how an exponent is fixed—this paper puts numbers on a case where the identity of one exponent turned out to be the dimension of space.Paper 144 read the exponent as the signature of what is conserved—the signature here is the dimension.Paper 190 measured rare on a logarithmic scale—this paper likewise measures an exponent by a log ratio.What is added is computing that the two models differ by a factor of 11814.7 at low temperature, exhibiting the difference between a power and an exponential by showing that the same measurement returns a meaningless value, solving in d dimensions and confirming that the exponent matches the dimension from 1.000137 to 3.999998, and writing that this dimension comes from the density of states and never appears in the quantisation.

1. Notation and Placement

Setting: compute the lattice specific heat of a solid in two models. **Both quantise the oscillator energy in integer multiples of $\hbar\omega$ —that much is shared.**

Model	Quantises?	Density of states $g(\omega)$
Einstein (1907)	yes	one and the same ω (a delta function)
Debye (1912)	yes	$\propto \omega^2$, from a linear dispersion

Question: then what separated the two?

The answer is the density of states alone. The quantisation is the same, and one line, $g(\omega)$, differs.

A caution: Debye in this paper means the lattice model. The 10 occurrences in earlier papers are the Debye length of a plasma, a different work by the same man—the two are not conflated.

2. At High Temperature They Cannot Be Told Apart

Setting: below, the specific heat is written as $C/(3Nk)$, a form approaching 1 in the high-temperature limit (Dulong and Petit).

Check 2.1: lower T/θ .

T/θ	Debye	Einstein	Ratio
1.00	0.951732	0.920674	1.0
0.50	0.825408	0.724062	1.1
0.20	0.368635	0.170742	2.2
0.10	7.582100×10^{-2}	4.540405×10^{-3}	16.7
0.05	9.740757×10^{-3}	8.244615×10^{-7}	11814.7

Observation 2.2

at high temperature the two models give the same answer. At $T/\theta = 1.00$, 0.951732 against 0.920674, a difference of some 3%—the two men cannot be told apart here.

Observation 2.3

lower T/θ by a factor of 20 and the ratio opens from 1.0 to 11814.7. They part company only at low temperature—whether the difference between models is visible depends on the temperature.

3. The Core: the Form of the Departure Differs

Setting: look at the form of the low-temperature behaviour. Debye is a power, T^3 ; Einstein is an exponential, $\exp(-\theta/T)$.

Check 3.1: take the two points $T/\theta = 0.02$ and 0.04 and measure an “exponent” by the log ratio.

Model	$T/\theta = 0.02$	$T/\theta = 0.04$	Measured exponent
Debye (three dimensions)	2.078061×10^{-4}	1.662448×10^{-3}	3.000000
Einstein	4.821875×10^{-19}	8.679965×10^{-9}	34.067

Observation 3.2

Debye comes out cleanly at 3.000000—because it is a power law. Einstein returns 34.067, a value with no meaning—it is not a power law, so the operation of measuring an exponent does not apply to it.

Observation 3.3

the 34.067 is a sign that the measurement was the wrong one. Take two other points and another value comes out— $\exp(-\theta/T)$ moves from $\exp(-50) = 1.93 \times 10^{-22}$ to $\exp(-25) = 1.39 \times 10^{-11}$, so the ratio depends on the temperature.

Observation 3.4

the separator—what has to be changed for these to become different things. Whether the density of states has states near $\omega = 0$. Debye has them, so however far one cools there remain low vibrations that can still be excited, and it falls as a power. Einstein has none, so below $\hbar\omega$ nothing can be excited at all, and it falls exponentially.

4. The 3 in T^3 Is the Dimension of Space

Setting: solve the Debye model in d dimensions. The dispersion stays linear and only the density of states changes, to $g(\omega) \propto \omega^{d-1}$.

Check 4.1: measure the exponent at the same two points.

Dimension d	$T/\theta = 0.02$	$T/\theta = 0.04$	Measured exponent
1	6.578486×10^{-2}	1.315822×10^{-1}	1.000137
2	2.884937×10^{-3}	1.153975×10^{-2}	2.000000
3	2.078061×10^{-4}	1.662448×10^{-3}	3.000000
4	1.990901×10^{-5}	3.185438×10^{-4}	3.999998

Observation 4.2

the measured exponent agrees with the dimension exactly. 1.000137, 2.000000, 3.000000, 3.999998—the 3 in T^3 is the dimension of space.

Observation 4.3

this is the chief point of the paper. The quantisation, the Planck distribution, contains d not once— d enters from the density of states alone. So saying that T^3 is a quantum effect says only half of it.

Observation 4.4

a point to state honestly—the small departure at $d = 1$, 1.000137, is because $T/\theta = 0.02$ is not yet cold enough there. It is a limit of the numerics, not a failure of the law.

5. A 4π Sits Inside That Density of States

Setting: in three-dimensional $\mathbf{k}$ -space, count the states within a radius k .

$$N(< k) = V \cdot \frac{\frac{4}{3}\pi k^3}{(2\pi)^3}, \quad g(\omega) d\omega = 3V \cdot \frac{4\pi k^2 dk}{(2\pi)^3}$$

Check 5.1: read off the constants standing in this expression.

Constant	Value	Origin
4π	12.566371	the whole solid angle of the sphere
$\frac{4}{3}\pi$	4.188790	the volume of the sphere
$(2\pi)^3$	248.050213	the k -space volume per state
3	3	the number of polarisations (one longitudinal, two transverse)

Observation 5.2

a 4π stands inside the density of states. It is the same thing as the 4π Paper 02 saw in an inverse-square field—both are the whole solid angle of the sphere, and they are not distinct roots.

Observation 5.3

Paper 258 wrote the condition for 4π to appear as three items, point source, three dimensions and isotropy. All three hold here—spheres in k -space are being counted isotropically about the origin, in three dimensions.

Observation 5.4

and in d dimensions the 4π becomes a different solid angle and k^2 becomes k^{d-1} . From that follow $g(\omega) \propto \omega^{d-1}$ and then $C \propto T^d$ —this one line is where the exponent measured in Section 4 comes from.

6. Roll Call and Closing

1. Both quantise (Section 1): what they share is the quantisation.
2. The same at high temperature (Check 2.1): **0.951732** and **0.920674**, ratio **1.0**.
3. A factor of **11814.7** at low (Check 2.1): at $T/\theta = 0.05$.
4. The forms differ (Check 3.1): **Debye a power, Einstein an exponential**.
5. **34.067** is meaningless (Observation 3.2): **the mark of measuring an exponent on a non-power**.
6. The separator is near $\omega = 0$ (Observation 3.4): **whether low vibrations remain**.

7. The exponent matches the dimension(Check 4.1): **1.000137, 2.000000, 3.000000, 3.999998.**

8. No d in the quantisation(Observation 4.3): d enters from the density of states.

9. A 4π sits there(Check 5.1): **12.566371, the whole solid angle.**

10. The same root as Paper 02(Observation 5.2): **not a distinct root.**

Closing: what separated Einstein and Debye was not quantisation.Both quantise exactly as much, and the difference is one line, the density of states $g(\omega)$.That one line makes a factor of **11814.7** at low temperature and changes even the form of the departure—Debye falls as a power and Einstein exponentially, so the same measurement applied to the latter returns a meaningless **34.067**.And the 3 in T^3 , measured by solving in d dimensions, comes out as **1.000137, 2.000000, 3.000000 and 3.999998**—the dimension of space itself.That dimension never appears in the quantisation and enters from the density of states alone.And inside that density of states sits $4\pi = 12.566371$, the same solid angle Paper 02 saw.One thing separates them—writing down which line it is that differs.Write it down, and what may be credited to quantisation separates from what must be credited to geometry.Do not write it down, and the 3 in T^3 is handed to quantum mechanics.

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