

What a Crystal Forbids Is Not the Symmetry but the Periodicity

— A Fivefold Rotation Returns $2 \cos(2\pi/5) = 0.618034$ — Remove the Translation and the Forbidden **5** Comes

Back — [Paper 269]

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— *From thinking to syncing.* —

Abstract

Fivefold symmetry, it is said, does not occur in crystals. This paper asks what forbids it—the answer is not the symmetry. What forbids it is the translation, and remove the translation and the **5** comes back. No new mathematical theorem and no new law is claimed. First, write the condition in integers. An n -fold rotation compatible with a lattice must have $2 \cos(2\pi/n)$ an integer (Section 2). Second, only five are allowed. $n = 1, 2, 3, 4, 6$ return $+2, -2, -1, 0, +1$ —and the integers are exhausted by these five (Section 2). Third, this is the core of the paper: $n = 5$ returns **0.618033989**—not an integer, and so forbidden (Section 3). Fourth, that value has a name. **0.618033989** is the reciprocal of the golden ratio, $1/\varphi$, itself, agreeing to within 1.11×10^{-16} (machine rounding) (Section 3). Fifth, what forbids it is the translation. Remove the translation and the fivefold symmetry returning **0.618033989** is allowed—quasicrystals and the Penrose tiling stand on exactly this number (Section 4). Sixth, a roll call. Wallpaper groups in two dimensions **17**, space groups in three **230**, crystallographic point groups **32**, Bravais lattices **14**—all finite, and not one of them contains a **5** (Section 5).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the crystallographic restriction theorem, that there are 17 wallpaper groups, 230 space groups, 32 crystallographic point groups and 14 Bravais lattices, and the existence of quasicrystals, are all standard. No crystallography is built—what is used is one cosine and a test for being an integer. No classification of groups is carried out—17, 230, 32 and 14 are cited values and were not counted here. The structure of quasicrystals is not treated—that fivefold symmetry is allowed without translation is written, and neither how real quasicrystals are made nor whether they are stable is treated. The Penrose tiling is not constructed—only its name is cited. Penrose is not confused with another—the 7 occurrences of Penrose in earlier papers (98, 100 and 103) are the physicist Roger Penrose and not the tiling. The golden ratio is not mystified—0.618034 appears because $2 \cos 72^\circ$ is a root of $x^2 + x - 1 = 0$, and nothing beyond that is read into it. Fivefold symmetry in three dimensions is not discussed—the argument concerns rotations compatible with a lattice, and molecular symmetry such as that of fullerene is not a matter of a lattice and is not treated. Relation to earlier papers: Paper 265 treated the density of states of a crystal—this paper looks at the same crystal from the side of symmetry. Paper 262 named the crystal as the bearer of yielding—this paper treats how things are arranged rather than how they break. Paper 201 counted “complete” as four different claims—this paper separates the phrase “a crystal forbids” into forbidding periodicity rather than symmetry. Paper 190 measured rare on a logarithmic scale—this paper writes by a discrete test, whether a number is an integer. What is added is computing $2 \cos(2\pi/n)$ for $n = 1$ to 12 and setting out the integer test, confirming that the value at $n = 5$ agrees with $1/\varphi$ to 1.11×10^{-16} , observing that $n = 10$ returns φ itself, and separating what is forbidden, the translation, from what is not, the symmetry.

1. Notation and Placement

Setting: place an n -fold rotation on a lattice, a set of points carried onto itself by translations. **If the rotation carries the lattice to the lattice, the two are said to be compatible.**

The condition: in a basis of the lattice the rotation is an integer matrix, so its trace forces $2 \cos(2\pi/n)$ to be an integer.

Question: why does fivefold symmetry not occur in a crystal?

The answer is not that the symmetry is forbidden. It is that it is incompatible with translation—remove the translation and the 5 is allowed.

A caution: Penrose in this paper means the tiling. The 7 occurrences in earlier papers are the physicist Roger Penrose, the same man but a different work—the two are not conflated.

2. Only Five Are Allowed

Check 2.1: compute $2 \cos(2\pi/n)$ for $n = 1$ to 12 and test whether it is an integer.

n	$2 \cos(2\pi/n)$	Verdict
1	+2.000000000	allowed
2	−2.000000000	allowed
3	−1.000000000	allowed
4	+0.000000000	allowed
5	+0.618033989	forbidden
6	+1.000000000	allowed
7	+1.246979604	forbidden
8	+1.414213562	forbidden
10	+1.618033989	forbidden
12	+1.732050808	forbidden

Observation 2.2

integers are returned by five values only, $n = 1, 2, 3, 4, 6$. For $n \geq 7$ it is never an integer again— $2 \cos(2\pi/n)$ rises monotonically towards 2, so past the +1 at $n = 6$ it stays trapped between 1 and 2.

Observation 2.3

the 5 is missing. 4 and 6 are allowed and only the 5 between them falls—a single hole in a run of consecutive numbers.

3. The Core: the Value Returned at $n = 5$ Has a Name

Check 3.1: set the value at $n = 5$ against the golden ratio.

Quantity	Value
$2 \cos(2\pi/5) = 2 \cos 72^\circ$	0.618033989
$1/\varphi = (\sqrt{5} - 1)/2$	0.618033989
Difference	1.11×10^{-16} (machine rounding)

Observation 3.2

a fivefold rotation returns the reciprocal of the golden ratio. The reason can be given algebraically— $2 \cos 72^\circ$ is a root of $x^2 + x - 1 = 0$ and φ a root of $x^2 - x - 1 = 0$, so they inhabit the same quadratic field.

Check 3.3: confirm the relation φ satisfies.

Quantity	Value
φ^2	2.618033989
$\varphi + 1$	2.618033989
$2 \cos(2\pi/10)$, returned at $n = 10$	$1.618033989 = \varphi$

Observation 3.4

$n = 10$ returns φ itself. The 5 and the 10 form a pair, $1/\varphi$ and φ —even the manner of their being forbidden is symmetric.

Observation 3.5

a point to state honestly—the golden ratio appears out of the algebra of the pentagon and means nothing more. No mystery is read into it here.

4. What Forbids It Is the Translation

Observation 4.1

where did the integer condition come from?From the single point that in a basis of the lattice the rotation is an integer matrix—and that holds because there is translational symmetry.

Check 4.2: set out what is allowed once the translation is removed.

Object	Translational symmetry	Fivefold symmetry
A crystal	present	forbidden
A quasicrystal	absent	allowed
The Penrose tiling	absent	allowed
A single molecule	absent	allowed

Observation 4.3

fivefold symmetry itself is forbidden nowhere.A pentagon can be drawn, and molecules with fivefold symmetry exist—what is forbidden is tiling the plane with it by translation.

Observation 4.4

the separator—what has to be changed for the 5 to return.Removing the translation.Neither the rotation, nor the dimension, nor the definition of symmetry has been moved.

Observation 4.5

so the phrase “a crystal forbids fivefold symmetry” has mistaken its subject.What forbids is not on the side of symmetry but on the side of periodicity.

5. A Roll Call

Check 5.1: set out what comes to a finite count in a crystal.

What is counted	Number
Wallpaper groups in two dimensions	17
Space groups in three dimensions	230
Crystallographic point groups	32
Bravais lattices in three dimensions	14
Allowed orders of rotation	5 of them (1, 2, 3, 4, 6)

Observation 5.2

every one is finite.And not one of them contains a fivefold rotation—not even among the 230.

Observation 5.3

that they come to a finite count is itself a consequence of the translation.On the side of quasicrystals, where the translation has been removed, there is no such finite classification—whether counting works at all is also settled by the translation.

6. Roll Call and Closing

1. The condition is integrality(Section 1): that $2 \cos(2\pi/n)$ be an integer.
2. Five are allowed(Check 2.1): $n = 1, 2, 3, 4, 6$.
3. Only the 5 is a hole(Observation 2.3): between 4 and 6.
4. $n = 5$ gives 0.618033989(Check 3.1): $1/\varphi$ to 1.11×10^{-16} .
5. $n = 10$ gives φ (Check 3.3): 1.618033989.
6. Integrality comes from translation(Observation 4.1): an integer matrix in a lattice basis.
7. Remove it and 5 is allowed(Check 4.2): quasicrystals and the Penrose tiling.

8. The subject was mistaken(Observation 4.5): what forbids is periodicity.

9. The roll call(Check 5.1): 17, 230, 32, 14, and no 5 in any of them.

Closing: the sentence “a crystal forbids fivefold symmetry” had mistaken its subject. What forbids is not on the side of symmetry but on the side of translation. The condition is that $2 \cos(2\pi/n)$ be an integer, and it comes from the single point that in a basis of the lattice a rotation is an integer matrix—which holds only because there is translational symmetry. At $n = 5$ the value returned is **0.618033989**, the reciprocal of the golden ratio, and at $n = 10$ it is φ itself. Remove the translation and quasicrystals and the Penrose tiling stand on that same 0.618033989—the pentagon could always be drawn; it simply could not be tiled. One thing separates them—writing down which side does the forbidding. Write it down, and the places where a 5 cannot appear separate from those where it may. Do not write it down, and when quasicrystals were found in 1984 one would have gone looking for the fault on the side of symmetry.

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