

Mean Field Does Not See the Dimension, and Yet the Dimension Alone Settles How Far It Errs

— In One Dimension There Is No Transition and It Says $kT_c/J = 2$; in Two It Is **76.27%** Too High — The Exponents Become Exact at $d \geq 4$ and T_c Never Does — [Paper 271]

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— *From thinking to syncing.* —

Abstract

The mean-field approximation predicts a phase transition in every dimension. This paper asks what settles how far it errs—the answer is the dimension alone. No dimension enters the approximation, and the size of the error is a function of the dimension nonetheless. No new mathematical theorem and no new law is claimed. First, in one dimension it errs without limit. Exactly, $T_c = 0$ and there is no transition; mean field says $kT_c/J = 2$ —no ratio can be formed (Section 2). Second, this is the core of the paper. The exact two-dimensional value (Onsager) is $kT_c/J = 2.269185$ and mean field gives **4**—**76.27%** too high (Section 3). Third, raising the dimension shrinks the error. **32.99%** in three dimensions and **19.76%** in four—falling monotonically (Section 3). Fourth, and yet it never reaches zero. The ratio runs $1.76275 \rightarrow 1.32993 \rightarrow 1.19755$ towards 1, and never attains it at any finite d (Section 4). Fifth, the exponents, on the other hand, become exact at $d \geq 4$. Above the upper critical dimension the critical exponents agree with the mean-field values—one and the same approximation becomes exact for one quantity and errs to the last for another (Section 5). Sixth, for continuous symmetry it is worse. By Mermin-Wagner there is no spontaneous breaking for $d \leq 2$, and mean field says it happens in every dimension (Section 6).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the mean-field approximation, Onsager’s exact solution, that the upper critical dimension is 4, and the Mermin-Wagner theorem are all standard. No statistical mechanics is built—what is used is one ratio and the comparison of four values. No exact solution is derived—Onsager’s $2/\ln(1 + \sqrt{2})$ is cited, and the three- and four-dimensional values are estimates from numerical work. The renormalisation group is not entered—Papers 117 and 120 treat it. This paper only sets out the size of the error. No critical exponent is computed—Paper 117 treats mean field as the naive dimensional counting that gives $\eta = 0$ and writes the upper critical dimension 4 from the power counting of the coupling. This paper does not enter that discussion of exponents and looks only at the value of T_c . Orders of limits are not treated—Paper 245 treats the order of limits in the Curie-Weiss model and states explicitly that it treats neither critical exponents nor the renormalisation group. This paper concerns the size of the error, not the order. Mermin-Wagner is not proved—only its conclusion is cited. The 8 occurrences of Mermin in earlier papers (11, 41, 185) are quantum measurement and the classification of defects, a different work by the same man. The upper critical dimension of percolation is not conflated with this—the 6 of Paper 264 belongs to percolation and the 4 here to the Ising model. Relation to earlier papers: Paper 117 separated the four ways in which scale invariance fixes an exponent—this paper looks at the transition temperature rather than the exponents. Paper 120 showed that dimensional analysis fails in two ways—this paper counts the size of the failure, dimension by dimension. Paper 245 showed that two limits have no answer until the order is written—this paper looks at the same mean-field model from the side of accuracy rather than order. Paper 264 wrote that the threshold belongs to the lattice and the exponents to the dimension—this paper writes that T_c and the exponents follow the dimension differently. The same shape of division. Paper 190 measured rare on a logarithmic scale—this paper writes in percentages and ratios. What is added is counting the error as 76.27%, 32.99% and 19.76% by dimension, writing that no ratio can be formed in one dimension, setting out $1.76275 \rightarrow 1.19755$ to show approach without attainment, and writing the asymmetry that the exponents become exact at $d \geq 4$ while T_c never does.

1. Notation and Placement

Setting: take the Ising model. Place a spin of ± 1 at each lattice site and sum $-J$ times the product over neighbouring pairs. On a simple cubic lattice in d dimensions the number of neighbours, the coordination number, is $z = 2d$.

The mean-field approximation: replace the field one spin feels by the average of its surroundings. The transition temperature then comes out as $kT_c/J = z = 2d$ —it sees only the number of neighbours.

Question: how far does this approximation err?

The answer is settled by the dimension alone. No “dimension” enters the approximation—only the coordination number does. And still the error is a function of the dimension.

A caution: Mermin in this paper means the Mermin-Wagner theorem. The 8 occurrences in earlier papers are quantum measurement and the classification of defects, a different work by the same man—the two are not

conflated.

2. In One Dimension It Errs without Limit

Check 2.1: solving the one-dimensional Ising chain exactly, **no magnetisation stands at any finite temperature.**

kT_c/J	
Exact (one dimension)	0 (no transition)
Mean field ($z = 2$)	2
Ratio	cannot be formed

Observation 2.2

this is not “poor accuracy”.It says that something is there which is not—a kind of error that cannot be written as a percentage.

Observation 2.3

the reason can be given in one line.In one dimension the cost of flipping a single place is finite, so order is destroyed at any temperature however low—and mean field does not see that “single place”.

3. The Core: Counting the Error by Dimension

Check 3.1: the exact two-dimensional value was given by Onsager (1944), $kT_c/J = 2/\ln(1 + \sqrt{2})$.

d	$z = 2d$	Mean field	Exact (or numerical)	Excess
1	2	2.0000	0.000000	cannot be formed
2	4	4.0000	2.269185	76.27%
3	6	6.0000	4.511500	32.99%
4	8	8.0000	6.680300	19.76%

Observation 3.2

the error shrinks monotonically.76.27 to 32.99 to 19.76 per cent—the higher the dimension, the better mean field does.

Observation 3.3

the separator—what has to be changed for the error to change.The dimension alone.Neither the strength of the coupling, nor the shape of the lattice, nor the size of the spin has been moved—and no dimension appears in the mean-field expression, while the error is a function of it.

Observation 3.4

the reason is that mean field sees no fluctuation at all.The higher the dimension the more neighbours there are, and fluctuations average away—the approximation improves not because it becomes right, but because what it ignored becomes small.

4. And Yet It Never Reaches Zero

Check 4.1: set out the ratio, mean field over exact.

d	Mean field over exact
2	1.76275
3	1.32993
4	1.19755
$d \rightarrow \infty$	1 (in the limit only)

Observation 4.2

it approaches 1 and never arrives at any finite d .Mean field is exact for T_c only in infinite dimension—in any real dimension it is out, however far one goes.

Observation 4.3

this says the same thing as the 76.27% of Section 3 in another notation.Both express “approaching without arriving”.

5. The Exponents Become Exact at $d \geq 4$ and T_c Never Does

Observation 5.1

the upper critical dimension is 4.For $d \geq 4$ the critical exponents agree with the mean-field values (up to logarithmic corrections).

Check 5.2: set the two quantities out by how they follow the dimension.

Quantity	At $d \geq 4$	At $d \rightarrow \infty$
Critical exponents	become exact	exact
Transition temperature T_c	still out by 19.76%	exact only there

Observation 5.3

one and the same approximation becomes exact for one quantity at a finite dimension and errs to the last for the other.The phrase “the dimension at which mean field becomes right” names two different things—4 if one means the exponents, and infinity if one means T_c .

Observation 5.4

Paper 117 treated the side of the exponents—placing mean field as the naive dimensional counting that gives $\eta = 0$, and writing the upper critical dimension 4 from the power counting of the coupling.This paper is on the side of T_c , and that the two sides give different answers is the content of this section.

Observation 5.5

Paper 264 wrote, for percolation, that the threshold belongs to the lattice and the exponents to the dimension. This paper writes, for the Ising model, that T_c and the exponents follow the dimension differently—the same shape of division appearing in another phenomenon.

6. For Continuous Symmetry It Is Worse

Observation 6.1

Ising spins take the two values ± 1 , that is, they are discrete. Make the spin continuously rotatable (Heisenberg or XY) and matters change.

Check 6.2: set the Mermin-Wagner conclusion beside the mean-field prediction.

Dimension	Spontaneous breaking of continuous symmetry	Mean field says
$d = 1$	does not occur	it occurs
$d = 2$	does not occur	it occurs
$d = 3$	occurs	it occurs

Observation 6.3

with discrete symmetry (Ising) order stands in two dimensions, and with continuous symmetry it does not. Mean field sees this distinction not at all—it returns the same answer either way.

Observation 6.4

so how far mean field errs depends not only on the dimension but on the type of the symmetry. The table of Section 3 is for discrete symmetry, and for continuous symmetry it is worse—a point to state honestly.

7. Roll Call and Closing

1. **Unbounded error in one dimension**(Check 2.1): **exactly 0, mean field 2.**
2. **An error no percentage can express**(Observation 2.2): **it says something is there which is not.**
3. **76.27% too high in two**(Check 3.1): **4 against 2.269185.**
4. **32.99% in three, 19.76% in four**(Check 3.1): **shrinking monotonically.**
5. **The separator is the dimension alone**(Observation 3.3): **which does not appear in the expression.**
6. **Approaching without arriving**(Check 4.1): **1.76275 $\rightarrow$ 1.19755.**
7. **The exponents are exact at $d \geq 4$** (Check 5.2): **T_c only at infinity.**
8. **The two sides differ**(Observation 5.3): **4, and infinity.**
9. **Worse for continuous symmetry**(Check 6.2): **it says order stands where none does.**

Closing: the mean-field expression contains no dimension at all. What it contains is the coordination number z —and still the error was a function of the dimension alone. In one dimension it says something is there which is not; in two it is 76.27% too high, in three 32.99%, in four 19.76%—shrinking monotonically and never reaching zero at any finite dimension. And the critical exponents do become exact at $d \geq 4$ —one and the same approximation becomes exact for one quantity at a finite dimension and only at infinity for the other. For continuous symmetry it is worse still, saying that order stands at $d \leq 2$ where none does. One thing separates them—writing down what one means by “mean field becomes right”. Write it down, and the 4 is seen to belong to the exponents and not to T_c . Do not write it down, and one reads mean field as having become entirely right in four dimensions.

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