

“Which Is Fastest” Has Three Answers, and a Different Method Wins Each

— Newton in **5** Iterations, the Secant Method in **7** Function Evaluations, Bisection on the First Step — Verified
by Actually Solving $x^3 - 2 = 0$ — [Paper 288]

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— *From thinking to syncing.* —

Abstract

Equations are solved numerically by Newton’s method, the secant method, or bisection. Asked which is fastest, one usually answers Newton. This paper asks whether the answer changes when “fast” is redefined—it changes three ways. No new mathematical theorem and no new law is claimed. First, we actually solve $x^3 - 2 = 0$. To reach 16 digits, Newton takes **5** iterations, the secant method **7**, bisection **53** (Section 2). Second, this is the core of the paper. Counted in total function evaluations, the secant method’s **7** beats Newton’s **10**—the verdict reverses (Section 3). Third, the efficiency index predicts it. $p^{1/m}$ is **1.414214** for Newton and **1.618034** for the secant method—the secant method is **14.41%** faster (Section 3). Fourth, there is a third reversal. In evaluations per digit on the first step, bisection’s **3.3219** beats Newton’s **3.6430** (Section 4). Fifth, the golden ratio stands here. The secant method’s order of convergence is the positive root of $p^2 = p + 1$, namely $\varphi = \mathbf{1.618034}$ (Section 5). Sixth, the separator is what counts as one unit. Iterations, function evaluations, or the first step—it is the convention of counting, not a property of the methods, that decides (Section 6).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—Newton’s quadratic convergence, the secant method’s order φ , bisection’s linear convergence, and the efficiency index $p^{1/m}$ are all standard. We do not build numerical analysis—all we use is one equation and three iterations. We do not prove the orders of convergence— $p = 2$ and $p = \varphi$ are merely quoted, and we do not enter the derivation of the error recursions. We assert nothing about general equations— $x^3 - 2 = 0$ is simple-rooted, smooth, and well conditioned, the most favourable case. At a multiple root quadratic convergence drops to linear, and with a poor starting value Newton diverges. We assert no values for the counts—the 5, 7 and 53 of Section 2 are measurements for this starting value, this equation, and this stopping criterion. Change the starting value and they move by ± 1 . We assume equal evaluation costs—we set the cost of f and f' equal. In practice f' may be dearer or cheaper, so the reversal of Section 3 depends on this assumption. We do not disparage bisection—bisection alone has guaranteed convergence. Comparisons of speed have meaning only on the premise that one converges at all. We assert nothing about the golden-ratio coincidence—the φ of Section 5 and the $1/\varphi$ of Paper 269 both come from $x^2 = x + 1$, but we do not claim any relation as phenomena. Relation to earlier papers: Paper 257 showed that changing the speed does not change the destination—this paper is the case where the definition of “speed” itself is threefold. Paper 255 separated two things called “accuracy,” only one removed by calibration—this paper likewise treats several quantities under one word. Paper 269 showed that a crystal’s five-fold symmetry is forbidden because $2 \cos 72^\circ = 1/\varphi$ —the φ here is a root of the same quadratic. Paper 159 showed that “ill posed” is not one word—this paper shows that “fast” is not one word. What is added is actually solving $x^3 - 2 = 0$ by three methods and counting iterations and evaluations, confirming that the verdict reverses under evaluation count, computing evaluations per digit at each stage and finding the third reversal in which bisection wins the first step, and recording that φ and $1/\varphi$ come from the same quadratic.

1. Three Methods and Three “Speeds”

We line up three iterative methods.

Method	Iteration	Function evaluations per step
Newton	$x \leftarrow x - f(x)/f'(x)$	2 (f and f')
Secant	$x \leftarrow x - \frac{f(x)(x - x_0)}{f(x) - f(x_0)}$	1 (f only)
Bisection	halve the interval	1

Observation 1.1

“Fast” admits three definitions.

Definition	What counts as one unit
(a) Iteration count	one iteration
(b) Total function evaluations	one call to the function
(c) Cost per digit	one digit of accuracy

Question: Is the winner the same under all three?

2. Actually Solving It: 5, 7, and 53

Setting: $x^3 - 2 = 0$, root $2^{1/3} = 1.259921049894873$. We iterate until the relative error falls below 10^{-16} .

Check 2.1: Newton (starting at $x = 1$).

Step	x	Correct digits
0	1.0000000000000000	0.686
1	1.3333333333333333	1.235
2	1.2638888888888888	2.502
3	1.2599334934499771	5.005
4	1.2599210500177698	10.011
5	1.2599210498948732	16.000

Observation 2.2

The digits run $0.686 \rightarrow 1.235 \rightarrow 2.502 \rightarrow 5.005 \rightarrow 10.011$, roughly doubling. This is quadratic convergence —the number of correct digits doubles each step.

Check 2.3: The secant method (starting at $x_0 = 1$, $x_1 = 1.5$).

Step	x	Correct digits
1	1.2105263157894737	1.407
2	1.2514085388756730	2.170
3	1.2602652758390120	3.563
4	1.2599187140887995	5.732
5	1.2599210492568176	9.295
6	1.2599210498948743	15.055
7	1.2599210498948732	16.000

Observation 2.4

The digits run $1.407 \rightarrow 2.170 \rightarrow 3.563 \rightarrow 5.732 \rightarrow 9.295 \rightarrow 15.055$, growing by about 1.6 each time. $2.170/1.407 = 1.542$, $3.563/2.170 = 1.642$, $9.295/5.732 = 1.622$ —oscillating about $\varphi = 1.618$.

Check 2.5: Bisection (interval $[1, 2]$).

Quantity	Value
Digits gained per step, $\log_{10} 2$	0.301030
Steps to reach 10^{-16} , $\log_2(1/(10^{-16} \cdot 2^{1/3}))$	52.8175 $\rightarrow$ 53

Observation 2.6

Under definition (a), iteration count, Newton wins with 5.Secant 7, bisection **53**—this is the ground for saying “Newton is fast”.

3. The Core: Counted in Evaluations, It Reverses

Setting: Newton evaluates two things per step, f and f' .Secant and bisection evaluate one.

Check 3.1:

Method	Iterations	Evaluations per step	Total evaluations
Newton	5	2	10
Secant	7	1	7
Bisection	53	1	53

Observation 3.2

This is the core of the paper. In iterations $5 < 7$ and Newton wins; in total evaluations $7 < 10$ and the secant method wins——on the same computation and the same run, the verdict swaps.

Check 3.3: We confirm with the efficiency index $p^{1/m}$ (p the order, m the evaluations per step).

Method	Order p	m	Efficiency index $p^{1/m}$
Newton	2.000000	2	1.414214
Secant	1.618034	1	1.618034
Bisection	1.000000	1	1.000000

Observation 3.4

The efficiency index predicts the observed reversal. $1.618034/1.414214 = 1.144123$, so the secant method is **14.41% faster**——the method with the lower order wins on efficiency.

Observation 3.5

A point to write honestly——this reversal depends on the assumption that f and f' cost the same. If f' is cheap analytically Newton wins; if f' is available only by numerical differentiation the secant method wins by more——the answer is decided not by the problem but by the cost table.

4. The Third Reversal: Bisection Wins the First Step

Setting: We compute definition (c), evaluations per digit, stage by stage.

Check 4.1: Newton (1 iteration = 2 evaluations).

Stage	Digits gained	Evaluations per digit
0 → 1	+0.549	3.6430
1 → 2	+1.267	1.5785
2 → 3	+2.503	0.7990
3 → 4	+5.006	0.3995
4 → 5	+5.989	0.3339

Check 4.2: The secant method (1 iteration = 1 evaluation).

Stage	Digits gained	Evaluations per digit
0 → 1	+0.687	1.4556
1 → 2	+0.763	1.3106
2 → 3	+1.393	0.7179
3 → 4	+2.169	0.4610
4 → 5	+3.563	0.2807
5 → 6	+5.760	0.1736

Check 4.3: Bisection is the same at every stage.

Stage	Digits gained	Evaluations per digit
every one	+0.3010	3.3219

Observation 4.4

We compare the first step.

	First step	Final stage
Newton	3.6430	0.3339
Secant	1.4556	0.1736
Bisection	3.3219	3.3219

Observation 4.5

Looking only at the first step, bisection's 3.3219 is cheaper than Newton's 3.6430. Superlinear methods are slow at first and abruptly fast later—the third reversal.

Observation 4.6

Bisection's column holds the same number top and bottom. Linear convergence means the cost per digit never changes, favourable at first and hopeless at the end—at the final stage it costs **9.95** times Newton and **19.14** times the secant method.

Observation 4.7

A point to write honestly—only the secant method's last stage ($6 \rightarrow 7$) is poor at 1.0582. That is because machine precision was reached and there was no headroom left, not a property of the method—a column whose significant figures are exhausted must not be used as evidence about speed.

5. Why the Golden Ratio Stands Here

Observation 5.1

The secant method’s error satisfies $e_{n+1} \sim C e_n e_{n-1}$. Substituting $e_n \sim A e_{n-1}^p$, the exponent must satisfy $p^2 = p + 1$ —whose positive root is $\varphi = (1 + \sqrt{5})/2 = 1.618033988749895$.

Observation 5.2

Paper 269 wrote that a crystal forbids five-fold symmetry because $2 \cos 72^\circ$ is not an integer. That value is **0.618033988749895**, that is $1/\varphi$ —the numerical difference is 1.11×10^{-16} , agreement at machine precision.

Check 5.3:

Source	Value	Equation satisfied
Order of the secant method	$\varphi = 1.618033988749895$	$p^2 = p + 1$
Crystal five-fold (Paper 269)	$2 \cos 72^\circ = 0.618033988749895$	$x^2 = -x + 1$ ($x = 1/\varphi$)

Observation 5.4

A point to write honestly—this paper claims no significance for the coincidence. Both are roots of quadratics with integer coefficients, and such numbers are few—the same number standing in two places is not the same root (the distinction of Paper 112).

6. Separator: What Counts as One Unit

Observation 6.1

Separator—what must be moved for the winner to change? The definition of one unit.

Definition of “fast”	Winner	Value
(a) Iteration count	Newton	5
(b) Total function evaluations	Secant	7
(c) Per digit on the first step	Bisection	3.3219

Observation 6.2

Under three definitions, three different methods win. All three are correct——what changed is not the methods but the counting.

Observation 6.3

Paper 159 showed that “ill posed” is not one word. This paper shows that “fast” is not one word——saying “fast” without naming a unit is like saying “large” without naming a number.

Observation 6.4

And there is a fourth axis——whether it converges at all. Bisection alone always converges given a sign-changing interval——Newton diverges for some starting values. Comparisons of speed have meaning only on the premise that one converges.

7. Roll Call and Closing

1. Newton, **5** iterations (Check 2.1): **Digits double.**
2. Secant, **7** iterations (Check 2.3): **Digits grow by about 1.6.**
3. Bisection, **53** iterations (Check 2.5): **0.301** digits per step.
4. Total evaluations **10** against **7** (Check 3.1): **It reverses.**
5. Efficiency **1.414214** against **1.618034** (Check 3.3): **14.41% faster.**
6. First step **3.3219** < **3.6430** (Observation 4.4): **Bisection wins.**
7. Final stage **3.3219** against **0.3339** (Observation 4.4): **9.95** times the cost.

8. φ is the positive root of $p^2 = p + 1$ (Observation 5.1): **1.618033988749895**.

9. $2 \cos 72^\circ = 1/\varphi$ (Check 5.3): **Difference 1.11×10^{-16}** .

10. **Three definitions, three winners** (Observation 6.1): **The counting decides**.

Closing: “Which is fastest” has three answers, and a different method wins each. Actually solving $x^3 - 2 = 0$: in iterations Newton wins with **5, and in total function evaluations the secant method wins with **7**—because Newton needs two things, f and f' , per step. The efficiency index $p^{1/m}$ predicts it—Newton **1.414214**, secant **1.618034**, so the method with the lower order is **14.41%** faster. And there is a third reversal—on the first step alone, bisection’s **3.3219** is cheaper than Newton’s **3.6430**. Superlinear methods start slow, while bisection’s cost per digit never changes. At the final stage, that constancy becomes a factor of **9.95**. One thing separates them—what counts as one unit. Iterations, function evaluations, or the first step. All three are correct, and all three return different answers. Saying “fast” without naming a unit is like saying “large” without naming a number.**

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