

# Only the Logarithm Adds

— +50% Followed by −50% Is Not 0% but −25% — Repeating ±10% Fifty Times Each Gives −39.50%,  
While the Naive Sum Answers 0% — [Paper 292]

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August 2026

— *From thinking to syncing.* —

## Abstract

A “rate of return” looks like a quantity that can be added. This paper asks which way of writing it can—the answer is the log return alone. No new mathematical theorem and no new law is claimed. First, we check on the smallest example. +50% then −50% sums to 0% in simple returns, but is actually −25% (Section 2). Second, this is the core of the paper. Exponentiating the sum of log returns  $\ln 1.5 + \ln 0.5 = -0.287682$  gives 0.750000—the difference from the measured value is 0 (Section 2). Third, “average” names three different operations. Arithmetic mean 0%, geometric mean −13.3975%, exponentiated log mean −13.3975% (Section 3). Fourth, the gap is approximated by  $\sigma^2/2$ . At ±50% it is 12.5% against 13.3975%, an error of 7.18%; at ±1% the ratio is 1.000025 (Section 4). Fifth, it bites over long series. ±10% fifty times each gives −39.50%, while the naive sum answers 0% (Section 5). Sixth, annualisation is not addition either. A daily  $\sigma = 1\%$  is 15.8745% a year, and multiplying by 252 overstates it by 15.87 times (Section 6).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the relation between simple and log returns, the geometric mean, volatility drag, and the  $\sqrt{n}$  rule are all standard. We do not build financial theory—all we use is one logarithm and one square root. We do not forecast prices—future returns and expectations are not treated at all. Only the composition rule for a given series is treated. We do not enter the tails—Paper 190 treated heavy tails and large deviations. This paper assumes no distributional shape and asks only whether addition is permitted. We do not hide the premise of the  $\sqrt{n}$  rule—the  $\sqrt{252}$  of Section 6 assumes independence between periods. With correlation it fails. We do not say  $\sigma^2/2$  is exact—as Section 4 shows, it is off by 7.18% at  $\pm 50\%$ . It is an approximation for small fluctuations. We give no investment advice—which mean to use depends on the question. This paper only separates which one answers what. Relation to earlier papers: Paper 194 showed that the four means are one family with only one fence of equality—that paper concerns the structure of the inequality at one instant; this one concerns which is right when composing along time, the same four means as material with orthogonal questions. Paper 132 showed that the 2 in Ito’s lemma is not a dimension—the  $\sigma^2/2$  here is that correction term itself, and the convergence table of Section 4 shows it numerically. Paper 190 measured “rare” on a logarithmic scale—here too one can add only after moving to the logarithm. Paper 272 showed that one and the same “twice” opens by 6.70 on the stimulus side—this paper is likewise about on which scale one adds. What is added is confirming that exponentiating the sum of logs leaves a difference of 0, lining up the three “averages” numerically, building a table in which the ratio to  $\sigma^2/2$  converges to 1.000025 as the fluctuation shrinks, and computing that  $\pm 10\%$  fifty times each gives  $-39.50\%$ .

## 1. Notation and the Question

Setting: When a price moves from  $P_0$  to  $P_1$  there are two ways to write it.

| Name          | Definition            | Range               |
|---------------|-----------------------|---------------------|
| Simple return | $r = P_1/P_0 - 1$     | $[-1, \infty)$      |
| Log return    | $\ell = \ln(P_1/P_0)$ | $(-\infty, \infty)$ |

Question: Joining two periods, which of them adds?

### Observation 1.1

Prices compose by multiplication— $P_2/P_0 = (P_1/P_0)(P_2/P_1)$ . Take a logarithm and multiplication becomes addition—the answer is already written in the form. This paper confirms it numerically.

## 2. The Core: The Sum of Logs Leaves a Difference of 0

Check 2.1: +50% then −50%.

| Method                   | Computation               | Result              |
|--------------------------|---------------------------|---------------------|
| Sum of simple returns    | $+0.5000 + (-0.5000)$     | 0.00% [wrong]       |
| Actual factor            | $1.5 \times 0.5$          | $0.7500 = -25.00\%$ |
| Sum of log returns       | $+0.405465 + (-0.693147)$ | -0.287682           |
| Exponentiated            | $e^{-0.287682}$           | 0.750000 [right]    |
| Difference from measured |                           | 0                   |

### Observation 2.2

This is the core of the paper. Exponentiating the sum of logs leaves a difference of 0 (not even machine rounding)——not close, but the same.

### Observation 2.3

The sum of simple returns misses by 25 percentage points. The difference between 0% and −25% is not an error but a different operation——something that should have been multiplied was added.

### Observation 2.4

Note that log returns are asymmetric. +50% is  $\ln 1.5 = +0.405$ ; −50% is  $\ln 0.5 = -0.693$ ——the same “50%” differs by 1.71 times in logarithm. The downside is heavier because −100% corresponds to  $-\infty$ .

## 3. “Average” Names Three Different Operations

Check 3.1: For the same two periods we compute three “averages.”

| Average                     | Computation                 | Value                  |
|-----------------------------|-----------------------------|------------------------|
| Arithmetic (simple returns) | $(0.5 - 0.5)/2$             | <b>0.0000%</b>         |
| Geometric (actual growth)   | $\sqrt{1.5 \times 0.5} - 1$ | <b>-13.3975%</b>       |
| Mean of log returns         | $(-0.287682)/2 = -0.143841$ |                        |
| Exponentiated               | $e^{-0.143841} - 1$         | <b>-13.3975%</b>       |
| Difference of the last two  |                             | $1.11 \times 10^{-16}$ |

### Observation 3.2

The geometric mean and the exponentiated log mean are the same. The difference is  $10^{-16}$ , machine rounding—the geometric mean is the arithmetic mean of the logarithms.

### Observation 3.3

Only the arithmetic mean answers 0%. The gap is **13.3975** percentage points—and both are called “the average return”.

### Observation 3.4

Paper 194 showed that the four means are one family with only one fence of equality. That concerns the structure of the inequality at one instant—this paper asks which returns the right answer when composed along time. The same four means as material, with orthogonal questions.

### Observation 3.5

Separator—which is right depends on the question. For “the expectation of the next single period,” the arithmetic mean; for “what it came to after  $n$  periods,” the geometric mean—both correct, answering different questions.

#### 4. The Gap Is Approximated by $\sigma^2/2$ , Exactly in the Small-Fluctuation Limit

Setting: The gap between arithmetic and geometric means is called “volatility drag” and is approximated by  $\sigma^2/2$ .

Check 4.1: We check at  $\pm 50\%$ .

| Quantity                                      | Value                                           |
|-----------------------------------------------|-------------------------------------------------|
| Standard deviation of simple returns $\sigma$ | 0.500000                                        |
| $\sigma^2/2$                                  | 0.125000 = 12.5000%                             |
| Measured drag (arithmetic – geometric)        | 0.133975 = <b>13.3975%</b>                      |
| Ratio                                         | <b>1.0718</b> (the approximation errs by 7.18%) |

Check 4.2: With the fluctuation set to  $\pm x$ , we watch the ratio.

| $\pm x$ | Measured drag | $\sigma^2/2 = x^2/2$ | Ratio           |
|---------|---------------|----------------------|-----------------|
| 0.500   | 0.13397460    | 0.12500000           | 1.071797        |
| 0.200   | 0.02020410    | 0.02000000           | 1.010205        |
| 0.100   | 0.00501256    | 0.00500000           | 1.002513        |
| 0.050   | 0.00125078    | 0.00125000           | 1.000626        |
| 0.020   | 0.00020002    | 0.00020000           | 1.000100        |
| 0.010   | 0.00005000    | 0.00005000           | <b>1.000025</b> |

**Observation 4.3**

Shrinking  $x$  by fifty takes the ratio from 1.0718 to **1.000025**.  $\sigma^2/2$  becomes exact in the small-fluctuation limit——this is the shape of Ito’s correction term.

#### Observation 4.4

Paper 132 showed that the 2 in Ito's lemma is not a dimension. This table shows numerically where that 2 appears—the 2 in  $x^2/2$  comes from the second-order term of a Taylor expansion and has nothing to do with the dimension of space.

#### Observation 4.5

A point to write honestly—at  $\pm 50\%$ ,  $\sigma^2/2$  is off by 7.18%. Applied to a series containing crashes it systematically understates—one must not forget it is an approximation.

## 5. It Bites over Long Series

Check 5.1:  $+10\%$  and  $-10\%$ , fifty times each, 100 periods in all.

| Method                                 | Result                        |
|----------------------------------------|-------------------------------|
| Sum of simple returns                  | 0% [wrong]                    |
| Actual factor                          | <b>0.60500607 = -39.4994%</b> |
| Sum of logs <b>50(ln 1.1 + ln 0.9)</b> | -0.50251679                   |
| Exponentiated                          | <b>0.60500607</b> [right]     |
| Geometric mean per period              | -0.501256%                    |
| $\sigma^2/2$ (per period)              | 0.500000%                     |

#### Observation 5.2

Over 100 periods it is **-39.50%**. The sum of simple returns answers 0%—39.5 percentage points of loss vanish entirely from view.

### Observation 5.3

The per-period geometric mean  $-0.501256\%$  agrees with  $\sigma^2/2 = 0.500000\%$  to three digits.  $\pm 10\%$  is small, so the approximation works well—exactly as the table of Section 4 predicted.

### Observation 5.4

And this  $-39.5\%$  results from equal numbers of rises and falls. The intuition that “even odds cancel out” misses by 39.5 percentage points—the fluctuation itself is a cost.

## 6. Annualisation Is Not Addition Either

Setting: If the periods are independent, variances add. So standard deviations go as  $\sqrt{n}$ .

Check 6.1: From a daily  $\sigma = 1\%$ .

| Horizon | Trading days $n$ | $\sigma\sqrt{n}$ |
|---------|------------------|------------------|
| Week    | 5                | 2.2361%          |
| Month   | 21               | 4.5826%          |
| Year    | 252              | 15.8745%         |

### Observation 6.2

The annual figure is **15.8745%**, not 252%. Naively multiplying by 252 gives 252%, an overstatement by **15.8745** times—because  $252/\sqrt{252} = \sqrt{252}$ .

### Observation 6.3

Here too the question is what adds. What adds is the variance, not the standard deviation—for returns the logarithm adds; for dispersion the variance adds.

#### Observation 6.4

A point to write honestly—the  $\sqrt{n}$  rule assumes independence between periods. With correlation it fails—independence is a premise, not an observed fact.

#### Observation 6.5

Separator—what must be moved for addition to become permitted? The scale.

| Quantity   | Composition rule | The writing that adds |
|------------|------------------|-----------------------|
| Return     | multiplication   | logarithm             |
| Dispersion | sum of squares   | variance              |

#### Observation 6.6

In neither case does the original quantity add. Move to the scale that adds, add there, and come back—the same procedure by which Paper 190 measured “rare” logarithmically.

## 7. Roll Call and Closing

1. +50% then −50% is −25% (Check 2.1): The simple sum says 0%.
2. The sum of logs leaves difference 0 (Check 2.1): Not even rounding.
3.  $\ln 1.5$  against  $\ln 0.5$  is 1.71 (Observation 2.4): The downside is heavier.
4. Geometric mean −13.3975% (Check 3.1): The arithmetic mean is 0%.
5. Geometric = exponentiated log mean (Observation 3.2): Difference  $1.11 \times 10^{-16}$ .
6.  $\sigma^2/2$  errs by 7.18% at  $\pm 50\%$  (Check 4.1): An approximation.
7. Ratio 1.000025 at  $\pm 1\%$  (Check 4.2): The shape of Ito’s correction.
8. −39.4994% over 100 periods (Check 5.1): The sum answers 0%.
9. −0.501256% per period (Check 5.1): Three digits with  $\sigma^2/2 = 0.5\%$ .
10. Annual 15.8745% (Check 6.1): 252 overstates by 15.87 times.

Closing: Returns do not add. Only the logarithm adds. +50% then −50% sums to 0% in simple returns but is actually −25%, while exponentiating the sum of logs, −0.287682, gives 0.750000 with a difference of 0. The single word “average” also names three operations—arithmetic 0%, geometric −13.3975%, and the geometric mean is nothing but the arithmetic mean of the logarithms (difference  $10^{-16}$ ). That gap is approximated by  $\sigma^2/2$ , and shrinking the fluctuation from  $\pm 50\%$  to  $\pm 1\%$  takes the ratio from 1.0718 to 1.000025—the shape of Ito’s correction term, appearing in the numbers. And it bites over long series— $\pm 10\%$  fifty times each, equal numbers of rises and falls, gives −39.4994%, and the naive sum answers 0%. One thing separates them—whether that quantity composes by multiplication or by addition. If by multiplication, move to logarithms and add there. If dispersion, move to variances and add there. What is added without moving is neither a return nor an average.

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