

A Drum Has No Pitch Because of Its Boundary, Not Its Equation

— A String Gives Integer Ratios; a Circular Membrane Misses the Simple Intervals by **78** to **148** Cents —

“Special Functions” Are Special in Their Boundary, and **4π** Appears Because the Coordinates Are Spherical —

[Paper 299]

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August 2026

— *From thinking to syncing.* —

Abstract

A guitar string has a pitch; a drum does not. Both obey the same wave equation. This paper asks what differs—the answer is the shape of the boundary. No new mathematical theorem and no new law is claimed. First, a string gives integer ratios and a circular membrane irrational ones. The membrane’s ratios are 1.000000, **1.593341**, 2.135549, **2.295417** (Section 2). Second, this is the core of the paper. They miss the simple intervals the ear locks onto by **78** to **148** cents, comparable to or greater than a semitone (100 cents) (Section 2). Third, for a string the gap is zero. Exactly **$2/1$** , **$3/1$** , **$4/1$** (Section 2). Fourth, the equation is the same. Both are the Laplacian; only the boundary differs (Section 3). Fifth, eleven coordinate systems admit separation of variables. Cartesian gives exponentials, spherical Legendre, cylindrical Bessel—the boundary chooses the function (Section 3). Sixth, which is why **4π** appears only in spherical coordinates. Cylindrical gives **2π** ; Cartesian gives not even **π** (Section 4).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the eigenmodes of a circular membrane, the zeros of Bessel functions, and Eisenhart’s eleven separable coordinate systems are all standard. We do not build the theory of partial differential equations—all we use is five ratios of zeros and one logarithm. We do not solve the eigenvalue problem—the Bessel zeros are merely quoted. We do not discuss instruments—that a timpani is struck off-centre, and that the air load pulls its partials into approximate order, are mentioned and not treated. Only the ideal circular membrane is treated. We do not build music theory—the psychology of consonance and systems of temperament are not treated at all. The cent is a scale for reading ratios. “Simple interval” is a convention—restricting numerator and denominator to 6 is this paper’s own convention, and raising the limit shrinks the gaps. The claim is that no ratio is an integer, not the value in cents. We do not treat the inverse problem—Paper 102 treated Kac’s inverse spectral problem (whether the shape follows from the spectrum). This paper runs forward: the shape fixes the ratios of the spectrum. The direction is reversed. We do not assert the number eleven—Eisenhart’s classification is for the Helmholtz equation in three dimensions, and the count changes with dimension or equation. Relation to earlier papers: Paper 02 showed that a pure solid angle 4π appears in inverse-square fields—this paper views it from the side on which that 4π is the consequence of choosing spherical coordinates. Paper 258 showed that 4π appears only when the source is a point—this paper restates the same condition from the side of the coordinate system. Paper 249 showed that diffusion’s 4π and a field’s 4π share a root—this paper confirms that root from behind, by listing the coordinate systems in which no 4π appears. Paper 276 showed that the decibel is a ratio and not a quantity—the cent here is likewise a logarithmic scale for a ratio, a sibling. What is added is measuring the membrane’s ratios as 78–148 cents from the simple intervals, setting the string’s zero beside them, attributing the “specialness” of special functions to the boundary by way of the eleven coordinate systems, and placing, as its consequence, the origin of 4π in the choice of coordinates.

1. Notation and the Question

Fact: Pluck a guitar string and a pitch is heard. Strike a drum and none is.

And yet: both obey the same equation.

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Question: If the equation is the same, what differs?

Observation 1.1

Only the boundary. The string is a segment (ends fixed), the membrane a disc (rim fixed)—this paper descends from that one point all the way to the presence or absence of pitch.

2. The Core: Integer Ratios for a String, Irrational for a Membrane

Setting: The eigenfrequencies of a fixed string are proportional to n ($n = 1, 2, 3, \dots$). Those of a circular membrane with a fixed rim are proportional to the Bessel zeros $j_{m,n}$.

Check 2.1: We line up the ratios to the fundamental.

Mode	Zero $j_{m,n}$	Ratio to fundamental	A string would give
$j_{0,1}$	2.404826	1.000000	1
$j_{1,1}$	3.831706	1.593341	2
$j_{2,1}$	5.135622	2.135549	3
$j_{0,2}$	5.520078	2.295417	4
$j_{3,1}$	6.380162	2.653066	5

Observation 2.2

The two right-hand columns agree nowhere. The string gives 1, 2, 3, 4, 5, integers; the membrane gives 1.593341, 2.135549, not integers—the partials do not harmonise, so no pitch is heard.

Check 2.3: We turn “not heard” into a number. We measure how far each ratio lies from a simple interval the ear locks onto (numerator and denominator at most 6). A semitone is 100 cents.

Mode	Ratio	Nearest simple ratio	Gap
$j_{1,1}$	1.593341	$5/3 = 1.666667$	−77.9 cents
$j_{2,1}$	2.135549	$2/1 = 2.000000$	+113.5 cents
$j_{0,2}$	2.295417	$5/2 = 2.500000$	−147.8 cents
$j_{3,1}$	2.653066	$5/2 = 2.500000$	+102.9 cents

Observation 2.4

This is the core of the paper. The gaps run **78 to 148 cents**——comparable to or greater than a semitone (100 cents). Not slightly off, but wholly elsewhere.

Check 2.5: We set the string beside it.

String partial	Ratio	Gap
2 nd	$2/1 = 2.000000$	0.0 cents
3 rd	$3/1 = 3.000000$	0.0 cents
4 th	$4/1 = 4.000000$	0.0 cents
5 th	$5/1 = 5.000000$	0.0 cents

Observation 2.6

The string's gap is exactly zero. Not close but integer by definition——that a string's partials are called “harmonic” is not a metaphor; it is the integer ratio.

Observation 2.7

A point to write honestly——restricting “simple interval” to numerator and denominator **6** is this paper's convention. Raise the limit and any number can be approximated by a fraction——the claim is that no ratio is an integer, not the value in cents.

Observation 2.8

A real timpani does have a pitch. It is struck off-centre and the air load draws the partials together——that is, it makes a pitch by departing from the ideal circular membrane.

3. The Equation Is the Same; the Boundary Chooses the Function

Observation 3.1

String and membrane are the same Laplacian. What changed is only in which coordinates one separates variables—and that is fixed by the shape of the boundary.

Check 3.2: We line up what emerges in three coordinate systems.

Coordinates	Function that emerges	Name
Cartesian	e^{ikz}	exponential
Spherical	$P_l^m(\cos \theta)$	Legendre
Cylindrical	$J_m(kr)$	Bessel

Observation 3.3

The equation is the same in all three. Only the emerging function differs—“special functions” are special in their boundary, not in their equation.

Observation 3.4

The Helmholtz equation in three dimensions separates in **11** coordinate systems, by Eisenhart’s classification. Elliptic cylinder, parabolic cylinder, prolate and oblate spheroidal, conical, ellipsoidal, paraboloidal, and the rest—each with its own “special function”.

Observation 3.5

So a list of special functions is not a list of functions but a list of boundaries. To learn Bessel functions is to learn the cylinder as a boundary—nothing essential lies on the function’s side.

Observation 3.6

Paper 102 treated Kac’s inverse spectral problem (whether the shape follows from the spectrum).This paper runs forward—the shape fixes the ratios of the spectrum, the reverse direction.That paper asks whether one can hear the shape; this one shows that the shape erases the pitch.

4. Which Is Why 4π Appears Only in Spherical Coordinates

Observation 4.1

We write the consequence of Section 3 on the side of the constants.

Check 4.2:

Coordinates	What the boundary carries	Constant that emerges
Cartesian	a plane	neither π nor 4π
Cylindrical	a circumference	2π
Spherical	a solid angle	$4\pi = 12.5663706144$

Observation 4.3

Paper 02 showed that a pure solid angle 4π appears in inverse-square fields.This paper views it from behind — 4π appears because spherical coordinates were chosen.

Observation 4.4

And choosing them is not our convenience.If the source is a point, the boundary (the equipotential surface) is a sphere—when Paper 258 wrote “ 4π appears only when the source is a point,” it said the same thing from the side of the source.

Observation 4.5

Paper 249 showed that the 4π diffusion catches and the 4π by which a field spreads share a root. This paper confirms that root from behind, by listing the coordinate systems in which no 4π appears—cylindrical yields only 2π , and Cartesian yields nothing.

Observation 4.6

So 4π is not a property of the equation. No 4π is written in the Laplacian—it appears the moment the boundary is taken to be a sphere. That a drum has no pitch and that 4π appears are two showings of one and the same fact—both are fixed by the boundary.

5. Separator: The Boundary Sits Above the Equation

Observation 5.1

Separator—what must be moved for a pitch to appear? The shape of the boundary.

Boundary	Ratios of eigenvalues	Pitch
Segment (string)	1, 2, 3, 4, . . . (integers)	yes
Disc (membrane)	1, 1.593341, 2.135549, . . . (irrational)	no

Observation 5.2

The equation is the same in both rows. Only the boundary changed—the phenomenon of pitch comes from the shape, not the equation.

Observation 5.3

Paper 276 showed that the decibel is a ratio and not a quantity. The cent here is likewise a logarithmic scale for a ratio—both treat dimensionless numbers that look like quantities; they are siblings.

Observation 5.4

Finally, what this paper did within the corpus. “Special functions,” “ 4π ,” and “pitch” are none of them properties of the equation; all three are properties of the boundary—and one and the same separator divides all three.

Observation 5.5

A point to write honestly——this paper does not say the boundary decides everything. Change the equation (say, from the wave equation to the diffusion equation) and the same boundary gives a different answer——what can be said is only that, with the equation held fixed, the boundary decides the rest.

6. Roll Call and Closing

1. Membrane ratios **1.593341, 2.135549** (Check 2.1): **A string gives 2, 3.**
2. Gaps of **78--148 cents** (Check 2.3): **A semitone or more.**
3. The string’s gap is **0.0 cents** (Check 2.5): **Integer by definition.**
4. The same Laplacian (Observation 3.1): **Only the boundary differs.**
5. Three coordinates, three functions (Check 3.2): **Exponential, Legendre, Bessel.**
6. Eleven separable systems (Observation 3.4): **A list of boundaries.**
7. Spherical **4π** , cylindrical **2π** (Check 4.2): **Cartesian, nothing.**
8. No **4π** in the Laplacian (Observation 4.6): **The boundary summons it.**
9. Reverse of Paper 102 (Observation 3.6): **That one hears the shape.**
10. The cent is a scale for a ratio (Observation 5.3): **Sibling of Paper 276.**

Closing: A drum has no pitch because of its boundary, not its equation. String and membrane obey the same Laplacian, yet a string’s partials are **2, 3, 4, 5**, integer ratios with a gap of **0.0 cents**, while a circular membrane gives **1.593341** and **2.135549**, irrational ratios lying **78 to 148 cents** from the simple intervals the ear locks onto——comparable to or greater than a semitone (100 cents). Not slightly off, but wholly elsewhere. The same holds for “special functions”——**11** coordinate systems admit separation, and Cartesian yields exponentials, spherical Legendre, cylindrical Bessel. A list of special functions is a list of boundaries, not of functions. And the same reasoning reaches **4π** ——spherical coordinates yield **4π** , cylindrical **2π** , Cartesian nothing at all. No **4π** is written in the Laplacian; it appears the moment the boundary is taken to be a sphere. One thing separates them——the shape

of the boundary. That a drum has no pitch and that 4π appears in an inverse-square field are two showings of one and the same fact.

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