

The Formula Returns 4, and at That One Point the Proof Does Not Reach

— Heawood's Formula Is a Theorem for $g \geq 1$ and Merely Returns the Right Value at $g = 0$ — The Simplest Surface Was Settled **78** Years Later, and by Computer — [Paper 303]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

Abstract

Heawood's formula $\chi(g) = \lfloor (7 + \sqrt{1 + 48g})/2 \rfloor$ gives the chromatic number of a surface of genus g . Put in $g = 0$ and out comes **4**—the very value of the four-colour theorem. This paper asks whether that is then a proof of the four-colour theorem—the answer is no. No new mathematical theorem and no new law is claimed. First, the formula returns the right value at $g = 0$. $(7 + \sqrt{1})/2 = 4.000000$, and the floor is **4** (Section 2). Second, this is the core of the paper. It is a theorem for $g \geq 1$, and at $g = 0$ the proof does not reach—Heawood's argument fails at that one point (Section 3). Third, a gap of 78 years opened. $g = 1$ was settled in 1890, all $g \geq 1$ in 1968, and $g = 0$ in **1976** (Section 4). Fourth, the simplest surface was left for last. And it fell only after 1936 reducible configurations were checked by computer (Section 4). Fifth, the non-orientable side has one exception too. For the Klein bottle the formula returns **7**, and the truth is **6** (Section 5). Sixth, the separator is the value being right against the proof reaching it (Section 6).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed——Heawood’s formula, the Ringel–Youngs theorem, the four-colour theorem, and Franklin’s exception are all standard. We do not build topology——all we use is one square root and one floor. We do not prove the four-colour theorem——we do not enter the Appel–Haken proof nor its later simplifications. We do not reproduce Heawood’s argument——the derivation of the bound from Euler’s formula is merely quoted. We assert no count of reducible configurations——1936 is the number in the 1976 version, reduced to 633 in Robertson et al. (1997). The claim is that the quantity is beyond human reading, not the count itself. We do not argue about computer proof——whether it should be accepted is not treated. We record only that it was formalised in Coq in 2005. Non-orientable surfaces are not a principal object——the Klein bottle of Section 5 is placed to show that an exception exists. Colouring of non-orientable surfaces is not treated systematically. Relation to earlier papers: Paper 302 treated how “it must appear” does not say when——this paper treats the case where the value is known and the proof does not reach it, the gap running the other way. Paper 163 separated “a good code exists” from “here is a good code”——this paper separates “the value is this” from “why”. Paper 179 showed that “cannot be constructed” has distinct roots——the $g = 0$ here is only one argument failing to reach, and another argument did reach. Paper 112 counted “six distinct roots sharing one rhyme”——here $g = 0$ and $g \geq 1$ share the value of one formula while their grounds differ. What is added is confirming that the formula returns 4.000000 at $g = 0$, measuring the 78-year gap between the settlement of $g \geq 1$ and of $g = 0$, recording the inversion by which the simplest surface was left for last, and placing the separator at the value’s correctness against the proof’s reach.

1. Notation and the Question

Setting: Any map drawn on a closed orientable surface of genus g can be coloured with $\chi(g)$ colours. Heawood (1890) gave this.

$$\chi(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

Question: What comes out at $g = 0$ (the sphere, hence the plane)? And is that a proof of the four-colour theorem?

Observation 1.1

The first half of the answer is “4 comes out.” The second half is “it is not”——this paper treats that gap.

2. The Formula Returns 4 at $g = 0$

Check 2.1:

Genus g	$(7 + \sqrt{1 + 48g})/2$	$\chi(g)$	Surface
0	4.000000	4	sphere
1	7.000000	7	torus
2	8.424429	8	double torus
3	9.520797	9	triple torus
4	10.446222	10	
5	11.262087	11	
6	12.000000	12	
7	12.678780	12	
10	14.465856	14	

Observation 2.2

At $g = 0$, $1 + 48 \cdot 0 = 1$, $\sqrt{1} = 1$, and $(7 + 1)/2 = 4.000000$. No floor is even needed: exactly 4—the very value of the four-colour theorem.

Observation 2.3

Note that $g = 6$ and $g = 7$ both give 12. The floor collapses them to one value—there are steps where the genus rises and the chromatic number does not.

Observation 2.4

By value alone, $g = 0$ looks like the natural first term of the sequence. 4, 7, 8, 9, 10, 11, 12, ... runs on without a break—the break is not on the value's side but on the proof's.

3. The Core: A Theorem for $g \geq 1$; at $g = 0$ the Proof Does Not Reach

Observation 3.1

What Heawood showed in 1890 is an upper bound——“ $\chi(g)$ colours suffice.”The argument bounds the average degree from Euler’s formula, and goes through for $g \geq 1$.

Observation 3.2

The lower bound (that a map actually needing that many colours exists) was completed by Ringel and Youngs in 1968.With that, the formula became a theorem for $g \geq 1$.

Observation 3.3

This is the core of the paper.At $g = 0$ alone, Heawood’s argument does not give the bound 4——the same argument gets only down to 5 colours (the five-colour theorem).

Check 3.4: We line up how far each reached.

Bound from Heawood’s argument		True value
$g \geq 1$	$\chi(g)$ (as the formula)	$\chi(g)$ (agrees)
$g = 0$	5 (five-colour theorem)	4 (does not agree)

Observation 3.5

The difference is just 1.The same argument reaches 5, and the last 1 will not fall——and that 1 took 86 years (1890 to 1976).

Observation 3.6

So the formula’s standing splits in two.

Standing of $\chi(g)$	
$g \geq 1$	a theorem (upper and lower bounds proved)
$g = 0$	an expression returning the right value (verified by another proof)

Observation 3.7

One and the same expression changes standing at a single point—the figure by which Paper 16 wrote that the equals sign too has distinct roots, occurring here within the domain of one function.

4. 78 Years — The Simplest Surface Was Left for Last

Check 4.1:

Surface	Chromatic number	Settled	How
Torus $g = 1$	7	1890 (Heawood)	by hand
$g = 2$	8	1968 (Ringel--Youngs)	by hand
All $g \geq 1$	the formula	1968	by hand
Sphere $g = 0$	4	1976 (Appel--Haken)	by computer

Observation 4.2

The simplest surface fell last. From the torus (1890) to the sphere (1976) is 86 years; from the settlement of all $g \geq 1$ (1968) it is still 8—the order of difficulty runs opposite to the order of simplicity.

Observation 4.3

The reason is how much slack there is. The larger the genus, the more slack in the Euler characteristic, and the better the degree bound works—the sphere has the least slack and so the argument works least.

Check 4.4: And the manner of settling differed too.

Quantity	Value
Reducible configurations checked by Appel--Haken (1976)	1936
Simplification by Robertson et al. (1997)	633
Formalisation in Coq (Gonthier)	2005

Observation 4.5

Reading 1936 configurations through is effectively impossible for a person. Even reduced to 633 in 1997, it still cannot be read through——only in 2005, when a machine verified it, did it become true that someone had seen all of it.

Observation 4.6

A point to write honestly——this paper does not argue about computer proof. What is recorded is only that $g \geq 1$ fell by hand and $g = 0$ did not.

5. The Non-Orientable Side Has One Exception Too

Setting: There is a corresponding formula for non-orientable surfaces.

Check 5.1:

Surface	Formula returns	Truth
Projective plane	6	6 (agrees)
Klein bottle	7	6 (does not agree)
All others	as the formula	agrees

Observation 5.2

The Klein bottle is the only exception (Franklin, 1934).The formula returns 7; six colours suffice——here the value itself is wrong.

Observation 5.3

So there are two kinds of gap.

	Value	Proof
$g = 0$ (sphere)	right	does not reach
Klein bottle	wrong	—

Observation 5.4

They are different failures.On the sphere the formula is right and cannot say why; on the Klein bottle it is simply wrong——the single phrase “the formula fails” covers both.

6. Separator: The Value Being Right against the Proof Reaching It

Observation 6.1

Separator——what must be moved for $g = 0$ to take the same standing as $g \geq 1$?The argument.Neither the surface nor the formula needs changing——bring another argument (checking reducible configurations) and the value stays while the standing changes.

Observation 6.2

Paper 302 treated how “it must appear” does not say when.Here the gap runs the other way——there the value was unknown; here the value was known and the reason could not be given.

	Existence	Value
Paper 302 (Ramsey)	known	unknown
This paper ($g = 0$, 1890--1976)	known	known (but unexplained)

Observation 6.3

So a third column is needed—the ground. Paper 163 separated existence from construction; Paper 302 separated existence from quantity—this paper separates the value from its ground.

Observation 6.4

And a value can be right while its ground comes from elsewhere. The 4 at $g = 0$ is not a correct prediction by Heawood's formula but another proof arriving at the same number—at $g = 0$ the formula merely “happens to be right”.

Observation 6.5

A point to write honestly—we do not say the formula is right at $g = 0$ by accident. The 4 comes out from the same estimate of the Euler characteristic, and is not unrelated—what can be said is only that at $g = 0$ that estimate does not reach the last step.

7. Roll Call and Closing

1. 4.000000 at $g = 0$ (Check 2.1): No floor even needed.
2. $g = 6$ and $g = 7$ both 12 (Observation 2.3): The floor collapses them.
3. The bound at $g = 0$ stops at 5 (Check 3.4): A difference of just 1.
4. That 1 took 86 years (Observation 3.5): 1890 to 1976.
5. $g \geq 1$ settled in 1968 (Check 4.1): By hand.
6. $g = 0$ settled in 1976 (Check 4.1): By computer.
7. 1936 reducible configurations (Check 4.4): 633 by 1997.
8. Formalised in 2005 (Check 4.4): A machine saw all of it.

9. Klein bottle: 7 against 6 (Check 5.1): The only exception.

10. Two kinds of failure (Observation 5.4): The value, or the ground.

Closing: Put $g = 0$ into Heawood's formula and **4.000000** comes out—the very value of the four-colour theorem. And it is not a proof of that theorem. The same argument gives both bounds for $g \geq 1$ and makes the formula a theorem, yet at $g = 0$ it gets only down to 5 colours—a difference of just 1, and that 1 took 86 years. And the order inverted—the torus fell by hand in 1890, all $g \geq 1$ by hand in 1968, and only the simplest surface, the sphere, fell in 1976 by computer. The 1936 reducible configurations could not be read through by a person, nor could the 633 of 1997, and only in 2005, when a machine verified them, did it become true that someone had seen all of it. One thing separates them—a value being right, and a proof reaching that value. The two are independent. The 4 the formula returns at $g = 0$ is correct, and its correctness came from elsewhere. One and the same expression changes standing at a single point of its domain.

References

1. Heawood, P. J. (1890). Map-Colour Theorem. *Quarterly Journal of Pure and Applied Mathematics*, 24.
2. Franklin, P. (1934). A Six Color Problem. *Journal of Mathematics and Physics*, 13(1-4).
3. Ringel, G. and Youngs, J. W. T. (1968). Solution of the Heawood Map-Coloring Problem. *Proceedings of the National Academy of Sciences*, 60(2).
4. Appel, K. and Haken, W. (1977). Every Planar Map Is Four Colorable. *Illinois Journal of Mathematics*, 21(3).
5. Robertson, N., Sanders, D., Seymour, P. and Thomas, R. (1997). The Four-Colour Theorem. *Journal of Combinatorial Theory, Series B*, 70(1).
6. Gonthier, G. (2008). Formal Proof — The Four-Color Theorem. *Notices of the American Mathematical Society*, 55(11).
7. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16]. Zenodo. 10.5281/zenodo.21600471
8. Yamagishi, Y. (2026). *The Allotted Region and the Freedom Within* [Paper 112]. Zenodo. 10.5281/zenodo.21962834
9. Yamagishi, Y. (2026). “A Good Code Exists” and “Here Is a Good Code” Are Different Sentences [Paper 163]. Zenodo. 10.5281/zenodo.22092677
10. Yamagishi, Y. (2026). “Cannot Be Constructed” Has Distinct Roots [Paper 179]. Zenodo. 10.5281/zenodo.22098419

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)