

The Same Integer Comes Back from Three Different Counts

— Integrate the Curvature, Count the Zeros of a Vector Field, Count the Passes of a Function — A Height Function on a Torus Needs Four Critical Points and Cannot Have Three — [Paper 316]

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— *From thinking to syncing.* —

Abstract

Paper 67 read $m = 4\pi = 2\pi\chi(S^2)$ as Poincare--Hopf. This paper asks how many roads lead to χ —the answer is three, and none of them follows from the other two. No new mathematical theorem and no new law is claimed. First, three counts return the same integer. The integral of curvature, the zeros of a vector field, the critical points of a function (Section 2). Second, this is the core of the paper. A height function on a torus needs four critical points and cannot have three (Section 3). Third, the curvature integral cannot see size. Radius 1 or 6371 km, the integral is **12.566371** (Section 4). Fourth, on a torus the positive and negative cancel exactly. Numerically below 10^{-13} , i.e. zero (Section 4). Fifth, each of the three needs something different. A metric, a vector field, a function (Section 5). Sixth, the separator is what extra object is brought in. χ itself needs none of them (Section 6).

Scope of this paper (scope note): No new mathematical theorem and no new law is claimed—the Gauss–Bonnet theorem, the Poincare–Hopf theorem, Morse theory with its inequalities, and the hairy ball theorem are all standard. We do not build topology—all we use is one numerical integral and one count of critical points. We prove none of the three theorems—no proof is entered. We check numerically that they agree and set out what each of them requires. We do not treat higher dimensions—closed surfaces only. The Gauss–Bonnet–Chern generalisation is not entered. We do not construct homology—Betti numbers are quoted, and the proof of the Morse inequalities is not entered. We do not treat singularities—degenerate critical points and surfaces with boundary are not treated. Relation to earlier papers: Paper 67 read $m = 4\pi = 2\pi\chi(S^2)$ as Poincare–Hopf—this paper places the third road (Morse) beside it and lines all three up. Paper 128 showed $4\pi = 2\pi\chi$ holds in three dimensions and nowhere else—this paper holds the dimension still and moves the counting instead. Paper 224 wrote “flat wherever you touch it, and four pi in total”—Section 4 here puts that cancellation on a torus into a number. Paper 91 showed that curvature is intrinsic—the other side of Gauss–Bonnet needing a metric. Paper 300 showed that whether two things share a root is decidable—this paper applies those criteria to the three roads. Paper 236 showed one and the same “ $= 0$ ” returning five different collections—this paper runs the other way: three different counts returning one integer. What is added is tabulating the three counts across four surfaces, integrating the curvature of a torus numerically and confirming zero, moving the radius of a sphere and showing the integral does not move, and setting out what each road requires and adjudicating them as distinct roots.

1. Notation and the Question

Setting: Take an orientable closed surface. For genus g (the number of holes) the Euler characteristic is $\chi = 2 - 2g$.

Observation 1.1

Paper 67 read $m = 4\pi = 2\pi\chi(S^2)$ as Poincare–Hopf, and Paper 128 showed $4\pi = 2\pi\chi$ holds in three dimensions and nowhere else.

Question: How many roads, then, lead to χ ?

Observation 1.2

The answer is three—and not one of them follows from the other two by substitution.

2. Three Counts Return the Same Integer

Check 2.1: Across four surfaces.

Surface	Morse (min – saddle + max)	Poincare--Hopf	Gauss--Bonnet $\oint KdA$
Sphere	$1 - 0 + 1 = +2$	$+2$	$+12.566371 = 2\pi \cdot 2$
Torus	$1 - 2 + 1 = 0$	0	$0 = 2\pi \cdot 0$
Genus 2	$1 - 4 + 1 = -2$	-2	$-12.566371 = 2\pi \cdot (-2)$
Genus 3	$1 - 6 + 1 = -4$	-4	$-25.132741 = 2\pi \cdot (-4)$

Observation 2.2

All three columns agree. One is an integral of a continuous quantity, the other two are discrete counts——that an integral returns an integer at all is the first strange thing.

Observation 2.3

Paper 236 wrote that one and the same “= 0” returns five different collections. This paper runs the other way——three different counts return one and the same integer.

3. The Core: A Torus Cannot Have Three

Setting: Stand a torus up and take height as the function. A critical point is one where the tangent plane is horizontal.

Check 3.1:

Critical point	Count	Index
Minimum (the floor)	1	0
Saddles (inner top and bottom)	2	1
Maximum (the top)	1	2
Alternating sum	—	$1 - 2 + 1 = 0$

Observation 3.2

This is the core of the paper. Fewer than four is impossible—the Morse inequalities bound the count from below by the sum of the Betti numbers.

Check 3.3:

Surface	Betti numbers b_0, b_1, b_2	Lower bound $\sum b_i$
Sphere	1, 0, 1	2
Torus	1, 2, 1	4
Genus 2	1, 4, 1	6
Genus g	1, $2g$, 1	$2 + 2g$

Observation 3.4

However the function is chosen, a torus cannot be given three. Set it differently, tilt it, deform it smoothly—four are needed—this is a property of the surface, not of the function.

Observation 3.5

The alternating sum and the total are different quantities. The alternating sum is $\chi = 0$; the total is at least 4— χ being zero does not make the critical points go away. They cancel; they are not absent.

Observation 3.6

Paper 227 wrote that the same $\leq$ points at a wall you can touch and at one you cannot. The Morse inequality is a wall you can touch—— $2 + 2g$ is actually attained, by a suitable height function.

4. The Curvature Integral Cannot See Size

Check 4.1: On a sphere of radius R , $K = 1/R^2$ everywhere and the area is $4\pi R^2$.

R [km]	K [/km ²]	Area [km ²]	$\oint K dA$
1	1.000000	12.566371	12.566371
2	0.250000	50.265482	12.566371
6371	2.463683×10^{-8}	5.100645×10^8	12.566371

Observation 4.2

The radius cancels. The Earth and a marble both give $12.566371 = 4\pi$ ——the integral is not looking at size.

Check 4.3: Integrate numerically on a torus (tube radius $r = 1$, ring radius $R = 3$). $K = \cos u / \{r(R + r \cos u)\}$ and the area element is $r(R + r \cos u) du dv$.

Quantity	Value
Strips	4000
$\oint K dA$ (computed here)	below 10^{-13}
Theory $2\pi\chi$ (with $\chi = 0$)	0

Observation 4.4

The $r(R + r \cos u)$ of the area element cancels against the denominator of the curvature. What is left is the integral of $\cos u$ once round, which is exactly zero—the positive curvature outside and the negative inside cancel once weighted by area.

Observation 4.5

Paper 224 wrote “flat wherever you touch it, and four pi in total.” The torus is the other side of that—nowhere flat, and yet zero in total.

Observation 4.6

Moving R and r leaves it at zero. Fatten the doughnut or thin it, nothing changes—a quantity that does not move when the shape does is a topological one.

5. Each of the Three Needs Something Different

Check 5.1:

Count	What it brings in	Without it
Gauss--Bonnet	a metric (a way of measuring length)	no curvature is defined
Poincare--Hopf	a vector field with isolated zeros	there is nothing to count
Morse	a function with nondegenerate critical points	no index is defined

Observation 5.2

All three bring in something the surface does not itself carry. A metric, a vector field, a function are all laid on afterwards—and yet the answer does not depend on how they are laid.

Observation 5.3

This is three ways of seeing that χ is a topological invariant. Change the tool and the answer does not change —the same structure as Paper 91’s “curvature is intrinsic”, one level further out.

Observation 5.4

The hairy ball theorem falls out of the second road.

Surface	χ	A nowhere-zero vector field
Sphere	+2	cannot exist
Torus	0	exists
Genus 2	−2	cannot exist

Observation 5.5

The torus is the combable one. The reason is not its shape but an integer—if $\chi \neq 0$ then every vector field vanishes somewhere. A sphere is not uncombable because it is round.

6. Separator: What Extra Object Is Brought In

Observation 6.1

Separator—what extra object the counting requires. χ itself requires none of them—count the vertices, edges and faces of a triangulation and it is there.

Check 6.2: Applying Paper 300’s criteria to the three roads.

Criterion	Result
Is there a correspondence preserving identities?	no. No construction makes zeros out of curvature
Do the same quantities appear?	no. Metric, vector field, function
Does one follow from another?	no. Substitution does not carry across
Do the values agree?	yes. Always χ

Observation 6.3

Three noes and one yes. The mechanisms are distinct roots and only the value is shared—exactly as Paper 300 wrote: the same number appearing is not the same root.

Observation 6.4

Why, then, do they agree? Because all three read out the same topological invariant with different tools, not because the tools resemble each other—the reason for the agreement lies underneath, not alongside.

Observation 6.5

A point to write honestly—this paper does not claim there are only three. In higher dimensions the index theorem is a frame containing all three—we have counted three in the narrow setting of closed surfaces.

Observation 6.6

One last thing— χ being zero is not the same as nothing being there. A torus has four critical points and its curvature vanishes nowhere—only the total is zero. Not reading cancellation as absence is the discipline here.

7. Roll Call and Closing

1. Three columns agree (Check 2.1): An integral returns an integer.
2. The torus gives $1 - 2 + 1 = 0$ (Check 3.1): Four critical points.

3. **Three is impossible** (Observation 3.4): **The bound is $2 + 2g$.**
4. **Alternating sum is not the total** (Observation 3.5): **$\chi = 0$ with four present.**
5. **The radius cancels** (Check 4.1): **12.566371 at every R .**
6. **The torus gives exactly zero** (Check 4.3): **Below 10^{-13} .**
7. **Moving R and r changes nothing** (Observation 4.6): **Hence topological.**
8. **Each needs something different** (Check 5.1): **Metric, field, function.**
9. **Only the torus is combable** (Observation 5.5): **The reason is an integer.**
10. **Three noes, one yes** (Check 6.2): **Distinct roots, shared value.**

Closing: Paper 67 read $m = 4\pi = 2\pi\chi(S^2)$ as Poincare--Hopf. Three roads lead to χ —integrate the curvature, count the zeros of a vector field, count the critical points of a function. Across four surfaces all three columns agree, and one of them is an integral of a continuous quantity that nonetheless returns an integer. And a height function on a torus needs four critical points and cannot have three—the Morse inequalities bound the count below by the sum of the Betti numbers, $2 + 2g$, and setting it differently or tilting it changes nothing. This is a property of the surface, not of the function. The curvature integral cannot see size—radius 1 or 6371 km, it is **12.566371**, because R cancels. On a torus the positive and negative cancel exactly, the numerical integral falling below 10^{-13} —fatten the doughnut or thin it and it stays zero, and a quantity that does not move when the shape does is a topological one. One thing separates them—what extra object the counting requires. Gauss--Bonnet needs a metric, Poincare--Hopf a vector field, Morse a function—none of which the surface itself carries, and yet the answer does not depend on how they are laid on. Paper 300's criteria return three noes and one yes: distinct roots, with only the value shared—the reason for the agreement lies underneath, not alongside. One last thing— χ being zero is not the same as nothing being there. A torus has four critical points and its curvature vanishes nowhere; only the total is zero, and cancellation must not be read as absence.

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